Entropy of $C(K)-$valued operators and some applications

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Introduction

Given a bounded linear operator $T : E \to F$ the n^{th} *dyadic entropy number* of T is defined by

$$e_n(T) \ := \ \inf\{\varepsilon > 0 \ : \ \exists\, x_1, \ldots, x_{2^{n-1}} \in E \ \text{ such that } \ TB_E \subset \bigcup_{k=1}^{2^{n-1}} (x_k + \varepsilon B_F)\} \ .$$

It is a simple fact that T is compact if and only if $(e_n(T))$ tends to zero. The rate of decay can be interpreted as a 'degree of compactness'. Entropy numbers are considered, for example, because they are known to be very helpful for eigenvalue estimates due to the inequality

$$|\lambda_n(T)| \ \leq \ \sqrt{2}\, e_n(T) \ .$$

Here the eigenvalues $\lambda_n(T)$ of a operator $T : E \to E$ are ordered by non-increasing absolute values and counted according to their algebraic multiplicities.
Let $T : E \to F$ be an arbitrary operator and $I : F \to C((B_{F'}, \omega^*))$ be the canonical injection then we always have

$$e_n(IT) \leq e_n(T) \leq 2e_n(IT) \ .$$

Therefore if we are interested in the *asymptotic* behaviour of entropy numbers, we can reduce our considerations to $C(K)$-valued operators. Note that without loss of generality we always may assume that $(B_{F'}, \omega^*)$ is metrizable, if we estimate entropy numbers of compact operators asymptotically. Therefore, let us additionally suppose that K is a compact *metric* space. One may ask whether the rate of decay of the sequence $(e_n(T : E \to C(K))$ is influenced by that of $(\varepsilon_n(K))$. The answer is positive if we additionally have some smoothness assumptions on T, say 1-Hölder-continuity. Now the general problem we treat can be stated as:

How are the entropy numbers of an arbitrary 1-Hölder-continuous operator $T : E \to C(K)$ influenced by the entropy numbers of K?

A quick answer can be given with the help of a well-known inequality due to Carl (cf. Theorem 1.4 and the introduction of Chapter 3):

$$\sup_{k \leq n} \ k^{1/p}\, e_k(T) \ \leq \ c_p\, c_K\, \|T\|_1 \ \sup_{k \leq n} \ k^{1/p}\, \varepsilon_k(K) \ . \tag{1}$$

Here c_p denotes a constant only depending on $p \in (0, \infty)$ and $c_K := \frac{1}{\min\{1, \varepsilon_1(K)\}}$. Hence for positive and decreasing sequences (a_n) with $a_n \sim a_{2n}$, e.g. $a_n = n^{-1/p}$, we obtain that $\varepsilon_n(K) \preceq a_n$ implies

$$e_n(T) \;\leq\; c\,\|T\|_1\, a_n$$

for all $n \geq 1$ and some constant $c > 0$. However, it is known by Carl, Heinrich and Kühn (cf. [9, Th.1], [12, Th. 5.10.1] or [10, Th. 2.3]) that if E is a Hilbert space or if at least the dual E' is of type q, then $\varepsilon_n(K) \preceq n^{-1/p}$ implies

$$e_n(T) \;\preceq\; n^{-(1-1/q)-1/p}\,. \tag{2}$$

Therefore, given a Banach space E, several questions arise:

- *When does inequality (1) yield asymptotically optimal results for some 1-Hölder-continuous operator $T : E \to C(K)$?*

  We will see in Chapter 5 that, roughly speaking, the inequality produces asymptotically optimal results if and only if E is not B-convex.

- *Are there other inequalities similar to (1) which take into account the local structure of E, e.g. in terms of type and cotype, and moreover cover the results of Carl, Heinrich and Kühn and give new estimates if $(\varepsilon_n(K))$ does not asymptotically behave like $n^{-1/p}$?*

  This is our major aim. We prove several types of such inequalities in Chapter 3.

- *Provided we find such inequalities, are the arising estimates asymptotically optimal for some 1-Hölder-continuous operator $T : E \to C(K)$?*

  The answer is definitely yes, as will be shown in Chapter 5.

- *What local structure must E have, if the estimate of Carl, Heinrich and Kühn holds for E? What are necessary conditions on E, if we have inequalities of the type described in the second question?*

  These questions are also considered in Chapter 5. For instance we will show that for $1 < q < 2$ implication (2) just characterizes B-convex Banach spaces of weak cotype q'.

As mentioned above, the estimate (1) is based on a general inequality between entropy numbers and approximation quantities due to Carl. Cases are also known in which an inverse form of this inequality holds. As an application of our results on 1-Hölder-continuous operators we present such an inverse inequality for operators $T : E \to F$ provided that one of the involved spaces is a Hilbert space. The idea of our approach is essentially based on the observation that there is a canonical metric on $B_{F'}$ induced by T' such that the operator IT is 1-Hölder-continuous with respect to this metric. In other words:

Every compact operator shares its entropy numbers with a suitable 1-Hölder-continuous operator.

This latter fact was implicitly used by Carl and Edmunds in [8]. That also allows to make some remarks on the duality problem of entropy numbers at the end of this work.
Several other applications of the cited result of Carl, Heinrich and Kühn are known, e.g. entropy estimates for integral operators with so-called Hölder-continuous kernels or for operators defined by abstract kernels. Moreover, given a precompact subset A of a Banach space E we can use (1) and (2) to estimate the entropy numbers of the absolutely convex hull $\mathrm{co}A$ by those of A, as worked out by Carl, Kyrezi and Pajor in [7] and [10]. More precisely, we obtain by (1) that

$$\varepsilon_n(A) \ \preceq \ a_n \qquad \text{implies} \qquad e_n(\mathrm{co}A) \ \preceq \ a_n \tag{3}$$

for all positive, decreasing sequences (a_n) with $a_n \sim a_{2n}$. Furthermore, if E is of type q, Carl, Kyrezi and Pajor proved with the help of (2) that

$$\varepsilon_n(A) \ \preceq \ n^{-1/p} \qquad \text{implies} \qquad e_n(\mathrm{co}A) \ \preceq \ n^{-1/p-(1-1/q)} \tag{4}$$

They also showed for type q spaces that $\varepsilon_n(A) \preceq (\log(n+1))^{-1/p}$ implies

$$e_n(\mathrm{co}A) \ \preceq \ \begin{cases} n^{-(1-1/q)} \ (\log(n+1))^{(1-1/q)-1/p} & \text{if} \quad p < q' \\ n^{-1/p} & \text{if} \quad p > q' \ , \end{cases} \tag{5}$$

where $1/q' := 1-1/q$. However several questions similar to our program for 1-Hölder-continuous operators are to be solved:

- *What happens if we know that $\varepsilon_n(A)$ only essentially decreases like in one of the above cases, e.g. if we have $\varepsilon_n(A) \preceq n^{-1/p}\ (\log(n+1))^\gamma$ for some $\gamma \neq 0$?*

  We will consider this question in Chapter 4. More precisely we will answer the following question raised by Ball and Pajor:

  Are there inequalities between $\varepsilon_n(A)$ and $e_n(\mathrm{co}A)$ which both cover the above implications and give new results?

- *When are the new and the results mentioned above asymptotically optimal for some subset A of E?*

  Roughly speaking, it turns out that (3) is asymptotically optimal for some subset A if and only if E is not B-convex. The other implications as well as the new ones are shown to be asymptotically optimal whenever E has no better (weak) type than q. Of course we cannot expect more.

- *What are necessary conditions on E such that the implications (4) and (5) hold?*

  We show that (4) is equivalent to E being of weak type q for $q \in (1,2)$. Moreover, if (5) holds in E then the space must be at least of all type $q-\varepsilon$.

We point out that entropy numbers of convex sets are very important for some stochastic questions. For instance, universal Donsker classes can be investigated by entropy conditions. This was the reason for Dudley in [16] to prove a weaker form of (4).

This work is organized as follows: In Chapter 1 we introduce all the necessary notions and prove some preliminary facts. We start with basic notations for spaces, operators and sequences. Then we present the entropy numbers and some other approximation quantities. Next we introduce some basic concepts from the local theory of Banach spaces. We then consider the socalled duality problem of entropy numbers and finally we present entropy estimates for finite rank operators acting between certain Banach spaces.
Chapter 2 is devoted to a decomposition technique for 1-Hölder-continuous operators which plays a fundamental role for our main theorems.
These results concern entropy estimates for 1-Hölder-continuous operators and are presented and proved in Chapter 3. Some consequences and further generalizations are also discussed there.
In Chapter 4 we apply these results to the problem of estimating entropy numbers of convex hulls. We also show that the proven results are asymptotically optimal. Moreover, we characterize some local properties of Banach spaces by entropy estimates for convex hulls and discuss an interesting phenomenon which occurs in B-convex Banach spaces. Finally we give some additional remarks on generalizations and open questions.
The fifth and last chapter is devoted to three topics: Firstly we investigate local properties of Banach spaces in terms of entropy estimates for 1-Hölder-continuous operators. We then show that our results of chapter 3 are asymptotically optimal. Although both of these questions could also be considered in Chapter 3 we decided to treat the 'dual case' first since this proceeding avoids some technical problems. Finally we present another application of a result of chapter 3: We show how Carl's inequality can be inverted provided that one of the involved spaces is a Hilbert space.

Parts of this dissertation, in particular the main theorems of Chapter 3 and 4 will be published in [37] by the *Journal of Approximation Theory.*

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Einleitung

Für einen stetigen, linearen Operator $T : E \to F$ ist die *n-te dyadische Entropiezahl* durch

$$e_n(T) \ := \ \inf\{\varepsilon > 0 \ : \ \exists \, x_1, \ldots, x_{2^{n-1}} \in E \ \text{ mit } \ TB_E \subset \bigcup_{k=1}^{2^{n-1}} (x_k + \varepsilon B_F)\}$$

definiert. Wie sich leicht überprüfen läßt, konvergiert die Folge der Entropiezahlen von T genau dann gegen 0, wenn T kompakt ist. Die Konvergenzgeschwindigkeit kann in diesem Fall als ein ,,Grad der Kompaktheit" angesehen werden. Mit Hilfe von Entropiezahlen lassen sich beispielsweise Eigenwerte durch die Ungleichung

$$|\lambda_n(T)| \ \leq \ \sqrt{2} \, e_n(T)$$

abschätzen, wobei hier die Eigenwerte $\lambda_n(T)$ eines Operators $T : E \to E$ gemäß ihrer algebraischen Vielfachheit gezählt und bezüglich ihres absoluten Wertes sortiert sind. Bezeichnet $I : F \to C((B_{F'}, \omega^*))$ die kanonische Einbettung, so gilt für jeden Operator $T : E \to F$ die Ungleichung

$$e_n(IT) \leq e_n(T) \leq 2e_n(IT) \ .$$

Daher genügt es, $C(K)$-wertige Operatoren zu untersuchen, sofern wir ,,nur" am asymptotischen Verhalten von Entropiezahlen interessiert sind. Betrachten wir ferner nur kompakte Operatoren - diese sind für Entropiezahlen die einzigen interessanten - , so können wir ohne Beschränkung der Allgemeinheit zudem annehmen, daß $(B_{F'}, \omega^*)$ metrisierbar ist. Daher sei im folgenden K immer ein kompakter, *metrischer* Raum. Nun kann man sich fragen, ob und inwieweit die Entropiezahlen von $T : E \to C(K)$ durch die von K beeinflußt werden. Falls wir zusätzlich wissen, daß T gewisse ,,Glattheitseigenschaften" besitzt, beispielsweise 1-Hölder-Stetigkeit, so gibt es in der Tat solche Abhängigkeiten. Das Problem, mit dem wir uns beschäftigen werden, kann daher in allgemeiner Form wie folgt formuliert werden:

Wie werden die Entropiezahlen von 1-Hölder-stetigen Operatoren $T : E \to C(K)$ durch die Entropiezahlen von K beeinflußt?

Mit Hilfe einer bekannten Ungleichung von Carl (vgl. Theorem 1.4 und die Einleitung von Kapitel 3) kann zunächst die folgende ad-hoc-Abschätzung gegeben werden:

$$\sup_{k \leq n} \ k^{1/p} \ e_k(T) \ \leq \ c_p \ c_K \ \|T\|_1 \ \sup_{k \leq n} \ k^{1/p} \ \varepsilon_k(K) \ . \tag{1}$$

Die Konstante c_p hängt hierbei nur von $p \in (0,\infty)$ ab. Zudem ist $c_K := \frac{1}{\min\{1,\varepsilon_1(K)\}}$. Betrachten wir nun eine positive, monoton fallende Nullfolge (a_n) mit der Regularitätsbedingung $a_n \sim a_{2n}$, also z.B. $a_n = n^{-1/p}$, erhalten wir daher für eine Konstante $c > 0$ und alle $n \geq 1$:

$$e_n(T) \;\leq\; c\,\|T\|_1\, a_n\;,$$

sofern wir $\varepsilon_n(K) \preceq a_n$ wissen. Dies ist in vielen Fällen jedoch nicht scharf. So haben Carl, Heinrich und Kühn (vgl. [9, Th.1], [12, Th. 5.10.1] und [10, Th. 2.3]) beispielsweise gezeigt, daß wir für Hilberträume E, oder allgemeiner für Banachräume E, deren Dualraum vom Typ q ist, immer

$$e_n(T) \;\preceq\; n^{-(1-1/q)-1/p} \tag{2}$$

erhalten, sofern $\varepsilon_n(K) \preceq n^{-1/p}$ bekannt ist. Für einen vorgegebenen Banachraum E stellen sich daher die folgenden Fragen:

- *Wann ergeben sich durch Ungleichung (1) asymptotisch optimale Abschätzungen für die Entropiezahlen eines geeigneten 1-Hölder-stetigen Operators $T : E \to C(K)$?*

  Wir werden im Kapitel 5 zeigen, daß sich im wesentlichen nur für nicht B-konvexe Räume asymptotisch optimale Abschätzungen ergeben.

- *Gibt es zu (1) vergleichbare Ungleichungen, die auch die lokalen Eigenschaften des Raumes E berücksichtigen und aus denen sich sowohl das oben zitierte Resultat von Carl, Heinrich und Kühn als auch bisher unbekannte Abschätzungen ableiten lassen?*

  Der Beweis solcher Ungleichungen ist unser Hauptziel. Im dritten Kapitel werden wir drei Klassen von ihnen zeigen.

- *Lassen sich aus den dann bewiesenen Ungleichungen asymptotisch optimale Abschätzungen herleiten?*

  Wir werden im fünften Kapitel Operatoren $T : E \to C(K)$ konstruieren, für die sich die Abschätzungen als asymptotisch optimal herausstellen werden.

- *Was sind notwendige Bedingungen an E, um die Abschätzung (2), bzw. die eben beschriebenen Ungleichungen erhalten zu können?*

  Dieser Frage werden wir ebenfalls im fünften Kapitel nachgehen. Wir werden dort beispielsweise zeigen, daß für $1 < q < 2$ die Abschätzung (2) gerade B-konvexe Räume vom schwachen Typ q' charakterisiert.

Wie bereits erwähnt, ist die Ungleichung (1) eine direkte Folge einer allgemeineren Ungleichung zwischen Entropie- und Approximationszahlen. Ferner sind auch Situationen bekannt, in denen eine umgekehrte Ungleichung dieser Art gilt. Es wird sich nun in Kapitel 5 zeigen, daß wir mit Hilfe unserer oben beschriebenen Abschätzungen eine weitere solche umgekehrte Ungleichung für Operatoren $T : E \to F$ beweisen können, sofern einer der beteiligten Räume ein Hilbertraum ist. Die Idee unseres Beweises basiert dabei im wesentlichen auf der Beobachtung, daß mit Hilfe des dualen Operators T' eine Metrik auf $B_{F'}$ konstruiert werden kann, bzgl. derer der Operator IT 1-Hölder-stetig ist. Mit anderen Worten gilt:

Für jeden Operator gibt es einen 1-Hölder-stetigen Operator, dessen Entropiezahlen mit denen des ersten asymptotisch übereinstimmen.

Dieses wurde implizit schon von Carl und Edmunds in [8] genutzt. Es ermöglicht uns zudem ein paar einfache Bemerkungen zum Dualitätsproblem für Entropiezahlen.
Das oben zitierte Resultat von Carl, Heinrich und Kühn hat diverse Anwendungen. Beispielsweise können mit ihm Integraloperatoren oder allgemeiner, Operatoren, die mit Hilfe abstrakter Kerne definiert wurden, untersucht werden. Eine weitere Anwendung der Abschätzungen (1) und (2) besteht in der Möglichkeit, für eine präkompakte Menge $A \subset E$ die Entropiezahlen der absolut konvexen Hülle $\mathrm{co}A$ durch die Entropiezahlen der Menge A abzuschätzen. Dieser Ansatz wurde von Carl, Kyrezi und Pajor in [7] und [10] benutzt. Sie erhielten so mit Hilfe der Ungleichung (1) für alle positiven, monoton fallenden Nullfolgen (a_n) mit $a_n \sim a_{2n}$, daß

$$e_n(\mathrm{co}A) \preceq a_n \tag{3}$$

gilt, sofern $\varepsilon_n(A) \preceq a_n$ bekannt ist. Ist E sogar vom Typ q, so leiteten sie aus der Abschätzung (2) ab, daß

$$e_n(\mathrm{co}A) \preceq n^{-1/p-(1-1/q)} \tag{4}$$

gilt, sofern $\varepsilon_n(A) \preceq n^{-1/p}$ für $A \subset E$ vorausgesetzt wird. Ferner zeigten Carl, Kyrezi und Pajor für Räume E vom Typ q und Mengen $A \subset E$ mit $\varepsilon_n(A) \preceq (\log(n+1))^{-1/p}$, daß

$$e_n(\mathrm{co}A) \preceq \begin{cases} n^{-(1-1/q)} \, (\log(n+1))^{(1-1/q)-1/p} & \text{falls} \quad p < q' \\ n^{-1/p} & \text{falls} \quad p > q' \end{cases} \tag{5}$$

gilt, wobei q' durch $1/q' = 1 - 1/q$ definiert ist. Damit ergeben sich analog zu unserem Programm für 1-Hölder-stetige Operatoren die folgenden Fragen:

- *Wie kann die Folge $(e_n(\mathrm{co}A))$ abgeschätzt werden, wenn sich die Folge $(\varepsilon_n(A))$ nur im wesentlichen wie in einem der obigen Fälle verhält? Was passiert beispielsweise im Falle von $\varepsilon_n(A) \preceq n^{-1/p} \, (\log(n+1))^\gamma$ für ein $\gamma \neq 0$?*

  Dieser Frage werden wir im vierten Kapitel nachgehen. Genauer gesagt werden wir dort sogar die folgende Frage von Ball und Pajor positiv beantworten:

  Gibt es Ungleichungen zwischen $\varepsilon_n(A)$ und $e_n(\mathrm{co}A)$, aus denen sich sowohl die bekannten als auch neue Abschätzungen herleiten lassen?

- *Wann sind die auf diese Weise gewonnenen und die schon bekannten Abschätzungen asymptotisch optimal?*

  Es stellt sich heraus, daß die Abschätzung (3) genau für nicht B-konvexe Banachräume asymptotisch optimal ist. Ferner sind sowohl die Abschätzungen (4) und (5) als auch die neu hinzugewonnenen immer dann asymptotisch optimal, wenn E keinen größeren Typ als q hat. Natürlich ist dies auch notwendig für die Optimalität.

- *Was sind notwendige Bedingungen an den Raum E, damit Abschätzungen wie (4) und (5) überhaupt gelten können?*

  Wir werden zeigen, daß die Abschätzung (4) Räume E vom schwachen Typ q charakterisiert. Gilt (5), so muß E zumindest vom Typ $q-\varepsilon$ für alle $\varepsilon > 0$ sein.

Abschließend sei erwähnt, daß Entropiezahlen absolut konvexer Hüllen für stochastische Fragestellungen eine wichtige Rolle spielen, beispielsweise bei der Beschreibung universeller Donsker-Klassen. Dies war der Grund für Dudley, in [16] eine abgeschwächte Form von (4) zu beweisen.

Der Aufbau dieser Arbeit ist wie folgt: Im ersten Kapitel führen wir alle notwendigen Begriffe und Sätze ein und beweisen einige, für das weitere Vorgehen nützliche Aussagen. Zunächst beginnen wir dabei mit Notationen für bestimmte Räume, Operatoren und Folgen. Sodann definieren wir Entropiezahlen und einige andere Approximationsgrößen und stellen wohlbekannte Eigenschaften von ihnen zusammen. Anschließend führen wir wichtige Konzepte aus der lokalen Banachraumtheorie ein. Danach beschäftigen wir uns mit dem sogenannten Dualitätsproblem für Entropiezahlen und stellen schließlich einige Entropieabschätzungen für endlichdimensionale Operatoren zwischen bestimmten Banachräumen vor.
Das zweite Kapitel ist einer Zerlegungstechnik für 1-Hölder-stetige Operatoren gewidmet, die eine fundamentale Rolle für unsere Hauptergebnisse spielt.
Diese Resultate über Entropieabschätzungen für 1-Hölder-stetige Operatoren werden im dritten Kapitel vorgestellt und bewiesen. Zudem werden einige Folgerungen und weitergehende Verallgemeinerungen, sowie offene Fragen diskutiert.
Im vierten Kapitel werden diese Ergebnisse benutzt, um Entropieabschätzungen für absolut konvexe Hüllen zu beweisen. Zudem zeigen wir, daß die so erzielten Ergebnisse praktisch immer asymptotisch optimal sind. Ferner charakterisieren wir einige lokale Eigenschaften von Banachräumen durch Entropieabschätzungen für absolut konvexe Hüllen und diskutieren diesbezüglich ein interessantes Phänomen, das in B-konvexen Banachräumen auftritt. Abschließend stellen wir wieder weitergehende Verallgemeinerungen und offene Fragen vor.
Das fünfte und letzte Kapitel widmet sich drei verschiedenen Themen: Zuerst untersuchen wir die lokale Strukur von Banachräumen mit Hilfe von Entropieabschätzungen für 1-Hölder-stetige Operatoren. Sodann zeigen wir, daß die im dritten Kapitel erzielten Resultate asymptotisch optimal sind. Diese beiden Problemstellungen hätten

natürlich auch schon im dritten Kapitel behandelt werden können, wir entschieden uns jedoch, zunächst den ,,dualen Fall" zu betrachten, da auf diese Weise einige technische Probleme entfallen und die Idee klarer zum Vorschein kommt. Schließlich wenden wir ein Resultat aus dem dritten Kapitel an, um eine Umkehrung der Carl'schen Ungleichung für den Fall zu zeigen, daß einer der beteiligten Räume ein Hilbertraum ist.
Teile dieser Dissertation, insbesondere die Hauptresultate des dritten und vierten Kapitels, werden in [37] durch das *Journal of Approximation Theory* veröffentlicht.

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Chapter 1

Preliminaries

1.1 Basic notations

In this text E and F always denote some Banach spaces. Continuous, linear maps between Banach spaces we always call *operators* and we usually denote them by T or S. For the closed unit ball of some Banach space E we write B_E. The dual space of E is denoted by E'. We write $E \stackrel{1}{=} F$ or $E \stackrel{1}{\hookrightarrow} F$, if the Banach space E is isometrically isomorphic to F, resp. isometrically embedded into F.
Let A be an arbitrary set and $0 < p < \infty$. Then $\ell_p(A)$ denotes the Banach space of all p-summable families of real numbers $(\xi_t)_{t\in A}$ over A with norm

$$\|(\xi_t)\|_p \ := \ \left(\sum_{t\in A} |\xi_t|^p\right)^{1/p} .$$

Analogously, we write $\ell_\infty(A)$ for the Banach space of all bounded number families $(\xi_t)_{t\in A}$ over A with norm

$$\|(\xi_t)\|_\infty \ := \ \sup_{t\in A} |\xi_t| \ .$$

In the case of $A = \mathbb{N}$ we just write ℓ_p instead of $\ell_p(A)$. Moreover, ℓ_p^n denotes the n-dimensional counterpart of ℓ_p. Given a real number $p \in [1, \infty]$ we always let $p' := \frac{p}{p-1}$, i.e. $\frac{1}{p'} = 1 - \frac{1}{p}$. For $1 \le p < \infty$ we then have $\ell_p' \stackrel{1}{=} \ell_{p'}$ for example.
Let $x = (x_i)$ be a sequence of real numbers, then we denote the non-increasing rearrangement of x by $(s_n(x))$. For $0 < p < \infty$ and $0 < q \le \infty$ the Lorentz sequence space $\ell_{p,q}$ is defined by

$$\ell_{p,q} \ := \ \{x \mid (n^{1/p-1/q} s_n(x)) \in \ell_q\}$$

which is equipped with the quasi-norm $\|x\|_{p,q} := \left\|(n^{1/p-1/q} s_n(x))\right\|_{\ell_q}$. Given a compact metric space (K, d), we write $C(K)$ for the space of all continuous functions $f : K \to \mathbb{R}$

with the usual supremum norm

$$\|f\|_\infty \;:=\; \sup_{x\in K} |f(x)| \; .$$

For an operator $T : E \to C(K)$ the *modulus of continuity* $\omega(T,.)$ is defined by

$$\omega(T,\delta) \;:=\; \sup_{x\in B_E} \sup_{d(s,t)\le\delta} |Tx(s) - Tx(t)| \qquad\qquad (\delta > 0) \; .$$

An operator $T : E \to C(K)$ is called *α-Hölder-continuous,* $0 < \alpha \le 1$, if

$$|T|_\alpha \;:=\; \sup_{\delta>0} \frac{\omega(T,\delta)}{\delta^\alpha} < \infty \; .$$

In this case we write $\|T\|_\alpha := \max\{\|T\|, |T|_\alpha\}$. If (A,d) is a precompact metric space and $T : E \to \ell_\infty(A)$ is an operator, we define α-Hölder-continuity of T and $\|T\|_\alpha$ analogously. Finally for $\varphi \in C(K)$ we let

$$\operatorname{supp}\varphi \;:=\; \{t \in K : \; \varphi(t) \neq 0\} \; .$$

Since we are interested in the asymptotic behaviour of sequences, it is reasonable to make some conventions. For given sequences $(a_n), (b_n)$ we write $a_n \preceq b_n$, if there exists a constant $c > 0$ such that $a_n \le c\, b_n$ for all $n \ge 1$. Moreover, we write $a_n \sim b_n$, if $a_n \preceq b_n$ and $b_n \preceq a_n$. A function $f : [0,\infty) \to (0,\infty)$ is said to be *σ-controlled,* $0 \le \sigma < \infty$, if

$$a^{-\sigma} f(t) \;\le\; f(a\cdot t) \;\le\; a^\sigma f(t)$$

holds for all $a, t \ge 1$. Note that 0-controlled functions are constant on $[1,\infty)$ and $t \to t^\sigma$ is σ-controlled. One easily checks that if f is σ-controlled and g is ρ -controlled then fg is $(\sigma+\rho)$-controlled and $1/f$ is σ-controlled. Moreover, if additionally g is monotone increasing with $g(1) \ge 1$ then $f \circ g$ is $(\sigma\rho)$-controlled. Therefore we are able to produce a lot of σ-controlled functions with the help of the following simple but important example.

Example 1.1 *Let $c > 1$ and $g : [0,\infty) \to (0,\infty)$ be defined by $t \mapsto \log_2(ct+1)$. Then g is $\frac{1}{\ln c}$-controlled.*
To see this we let $\sigma := (\ln c)^{-1}$. Since $\frac{\partial}{\partial a}(ac^{1-a^\sigma}) = c^{1-a^\sigma}(1-a^\sigma) \le 0$ for all $a \ge 1$ we observe that $ac^{1-a^\sigma} \le 1$ for all $a \ge 1$. Hence for $a, t \ge 1$ we obtain

$$act \;=\; a(ct)^{1-a^\sigma}(ct)^{a^\sigma} \;\le\; ac^{1-a^\sigma}(ct)^{a^\sigma} \;\le\; (ct)^{a^\sigma} \; .$$

Therefore g is σ-controlled since for all $a, t \ge 1$ we have

$$a^{-\sigma} g(t) \le g(at) \;=\; \log_2(act+1) \;\le\; \log_2((ct)^{a^\sigma}+1) \;\le\; \log_2((ct+1)^{a^\sigma}) \;=\; a^\sigma\, g(t) \; .$$

A positive null sequence (a_n) is said to be *regular*, if there is a constant $c \geq 1$ such that $a_n \leq c\ a_{2n}$ and $a_m \leq c\ a_n$ for all $1 \leq n \leq m$. Note, that in this case we always have

$$a_n \ \leq \ c^2 \ \left(\frac{m}{n}\right)^{\log_2 c} \ a_m$$

for all $n \leq m$. With the help of σ-controlled functions we are able to construct regular sequences as the following example shows:

Example 1.2 *Let f be a σ-controlled function and $0 < p < \infty$. Then the sequences*

$$a_n := n^{-1/p}\ f(\log(n+1)) \qquad \textit{and} \qquad b_n := (\log(n+1))^{-1/p}\ f(\log(\log(n+1)+1))$$

are regular.

To see this for the sequence (a_n) we first observe that $a_n \leq n^{-1/p}\ (\log(n+1))^{\sigma}\ f(1) \to 0$ for $n \to \infty$. For $n \geq 1$ we also have

$$a_n \ = \ n^{-1/p}\ f(\log(n+1)) \ \leq \ 2^{1/p}\ (2n)^{-1/p}\ \left(\frac{\log(2n+1)}{\log(n+1)}\right)^{\sigma}\ f(\log(2n+1)) \ \leq \ 2^{1/p+\sigma} a_{2n}\ .$$

Moreover, there is a $c > 0$ such that $\left(\frac{n}{m}\right)^{1/p}\ \left(\frac{\log(m+1)}{\log(n+1)}\right)^{\sigma} \leq c$ for all $1 \leq n \leq m$. Hence we obtain

$$a_m \ = \ m^{-1/p}\ f(\log(m+1)) \ \leq \ \left(\frac{n}{m}\right)^{1/p}\ \left(\frac{\log(m+1)}{\log(n+1)}\right)^{\sigma}\ n^{-1/p}\ f(\log(n+1)) \ \leq \ c\ a_n$$

for all $1 \leq n \leq m$. The proof for the sequence (b_n) is analogous.

Finally we give an example of a very slowly decreasing regular sequence:

Example 1.3 *There exists a monotone decreasing regular sequence (a_n) with $a_n \sim a_{2^n}$*

For the construction of such a sequence we first define a sequence (b_n) inductively by $b_1 := 1$ and $b_{n+1} := 2^{b_n}$. Now we let

$$a_n := 2^{-k} \qquad\qquad \text{iff} \ \ b_k \leq n < b_{k+1}\ .$$

Of course (a_n) is a monotone decreasing sequence with $a_n \to 0$. Now for given $n \in \mathbb{N}$ there is a $k \in \mathbb{N}$ with $b_k \leq n < b_{k+1}$. We then have $b_{k+1} \leq 2^n < b_{k+2}$ and therefore

$$a_n \ = \ 2^{-k} \ = \ 2 \cdot 2^{-(k+1)} \ = \ 2\ a_{2^n}\ .$$

Moreover, since $2n \leq 2^n$ we also have $a_n = 2a_{2^n} \leq 2a_{2n}$, i.e. (a_n) is regular.

1.2 Entropy numbers and some s-numbers

Let (K, d) be a metric space and $B(x, \varepsilon) := \{y \in K \ : \ d(x, y) \leq \varepsilon\}$ be the closed ball with radius ε and centre x. Then for a bounded subset $M \subset K$ the n^{th} *entropy number* of M is defined by

$$\varepsilon_n(M) \ := \ \inf\{\varepsilon > 0 \ : \ \exists\ x_1, \ldots, x_q \in K,\ q \leq n \text{ such that } M \subset \bigcup_{k=1}^{q} B(x_k, \varepsilon)\}\ .$$

The n^{th} *dyadic entropy number* of M is

$$e_n(M) \ := \ \varepsilon_{2^{n-1}}(M) \ .$$

Given an operator $T : E \to F$ the n^{th} *dyadic entropy number* of T is defined by

$$e_n(T) \ := \ e_n(T(B_E)) \ .$$

Moreover, we are interested in the following numbers associated with T:

- the n^{th} *approximation number* of T, defined by

  $$a_n(T) \ := \ \inf\{ \ \|T - A\| \ : \ A : E \to F \text{ bounded, linear with rank } A < n \ \} \ .$$

- the n^{th} *Gelfand number* of T defined by

  $$c_n(T) := \inf\{ \ \left\|T_{|E_o}\right\| \ : \ E_o \text{ subspace of } E \text{ with codim } E_o < n \ \} \ .$$

- the n^{th} *Kolmogorov number* of T defined by

  $$d_n(T) := \inf\{ \ \left\|Q^F_{F_o} T\right\| \ : \ F_o \text{ subspace of } F \text{ with } \dim \ F_o < n \ \} \ ,$$

  where $Q^F_{F_o}$ denotes the canonical surjection from the Banach space F onto the quotient space F/F_o.

- the n^{th} *Tichomirov number* of T defined by

  $$t_n(T) := a_n(I_F T Q_E) \ ,$$

  where $I_F : F \to \ell_\infty(B_{F'})$ is the canonical embedding and $Q_E : \ell_1(B_E) \to E$ is the canonical surjection.

For $s \in \{e, a, c, d, t\}$ and arbitrary operators $T, S : E \to F$ the sequence $(s_n(T))$ is monotone decreasing and we have $s_1(T) = \|T\|$. For $n, m \in \mathbb{N}$ we also know

$$s_{n+m-1}(S + T) \ \leq \ s_n(S) + s_m(T) \ .$$

This and the following properties can be found in [12, Ch. 1 and 2] and [31, Ch. 2]. Moreover, for $T : E_1 \to E_2$, $S : E_2 \to E_3$ and $n \in \mathbb{N}$ we have

$$s_n(ST) \ \leq \ \|S\| \ s_n(T) \qquad \text{and} \qquad s_n(ST) \ \leq \ s_n(S) \ \|T\| \ .$$

Except for the Tichomirov numbers all of the above sequences are also *multiplicative* in the sense of

$$s_{n+m-1}(ST) \ \leq \ s_n(S) \ s_m(T) \ .$$

For $s \in \{e, d, t\}$ these numbers are *surjective*, i.e. $s_n(TQ) = s_n(T)$ for every metric surjection Q. For $s \in \{c, t\}$ these numbers are *injective*, i.e. $s_n(IT) = s_n(T)$ for every metric injection I. The entropy numbers are only injective in the weaker sense of $e_n(IT) \le e_n(T) \le 2e_n(IT)$ for every metric injection I.
The s-numbers are ordered by

$$t_n(T) \le c_n(T) \le a_n(T) \qquad \text{and} \qquad t_n(T) \le d_n(T) \le a_n(T) \ .$$

Moreover, for $T : E \to F$ we have $a_n(T) = c_n(T)$, resp. $a_n(T) = d_n(T)$ if E, resp. F is a Hilbert space (cf. [12, Prop. 2.4.1. and 2.4.4.]). The following theorem is due to Carl (cf. [3, Th. 1]) and gives a relation between entropy numbers and the above quantities. Another, more elementary proof can be found in [12, Th. 3.1.1.].

Theorem 1.4 *For every $0 < p < \infty$ there is a constant $c_p \ge 1$ such that for every operator $T : E \to F$ and all $n \in \mathbb{N}$ we have*

$$\sup_{k \le n} k^{1/p}\ e_k(T) \ \le\ c_p\ \sup_{k \le n} k^{1/p}\ t_k(T)\ .$$

Let $s \in \{e, a, c, d, t\}$, $0 < p < \infty$ and $0 < q \le \infty$. Then for Banach spaces E and F we define

$$\mathcal{L}^{(s)}_{p,q}(E, F) := \{T : E \to F\ :\ T \text{ linear and continuous with } (s_n(T)) \in \ell_{p,q}\}\ .$$

Equipped with the quasi-norm $\lambda^{(s)}_{p,q}(T) \ :=\ \|(s_n(T))\|_{p,q}$ this set becomes a quasi-Banach space. In particular, for all $T_1, \dots, T_n \in \mathcal{L}^{(s)}_{p,\infty}(E, F)$ we have

$$\lambda^{(s)}_{p,\infty}\Big(\sum_{i=1}^{n} T_i\Big) \ \le\ c_p \left(\sum_{i=1}^{n} \big(\lambda^{(s)}_{p,\infty}(T_i)\big)^r \right)^{1/r},$$

where $r = \frac{p}{1+p}$ and $c_p > 0$ is a constant only depending on p.
Finally we state some well-known facts on certain entropy numbers. The following lemma can be found in [12, p.9]:

Lemma 1.5 *Let E be an $n-$dimensional real Banach space. Then we have*

$$2^{-\frac{k-1}{n}} \ \le\ e_k(B_E) \ \le\ 4 \cdot 2^{-\frac{k-1}{n}} \qquad\qquad \textit{for all } k \ge 1.$$

The next theorem is due to Schütt (cf. [36, Th. 1]):

Theorem 1.6 *Let $1 \le p < q \le \infty$. Then there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all $\log n \le k \le n$ we have*

$$c^{-1} \left(\frac{\log_2(\frac{n}{k} + 1)}{k} \right)^{1/p - 1/q} \ \le\ e_k(id : \ell^n_p \to \ell^n_q) \ \le\ c \left(\frac{\log_2(\frac{n}{k} + 1)}{k} \right)^{1/p - 1/q} .$$

1.3 Structures of finite dimensional subspaces

Given two n-dimensional Banach spaces E and F the *Banach-Mazur-distance* between E and F is defined as

$$d(E,F) \ := \ \inf\{\|T\| \ \left\|T^{-1}\right\| \ \mid T : E \to F \text{ isomorphism } \} \ .$$

The next definition describes Banach spaces whose finite dimensional subspaces look like the spaces ℓ_p^n:

Definition 1.7 Let $1 \le p \le \infty$ and $\lambda > 1$. A Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-*space* if every finite dimensional subspace E of X is contained in a finite dimensional subspace F of X such that $d(F, \ell_p^{\dim F}) < \lambda$.
We say that X is an $\mathcal{L}_p$-*space* if it is an $\mathcal{L}_{p,\lambda}$-space for some $\lambda > 1$.

For μ being a measure and $1 \le p \le \infty$ the Lebesgue space $L_p(\mu)$ is an $\mathcal{L}_p$-space. Moreover, for K being a compactum the space $C(K)$ is an $\mathcal{L}_\infty$-space. For a proof see e.g. [15, Th. 3.2].
In the following definition we describe the Banach spaces which contain arbitrarily large, finite dimensional subspaces being close to the corresponding ℓ_p^n's.

Definition 1.8 Let $1 \le p \le \infty$ and $\lambda > 1$. We say that the Banach space E *contains ℓ_p^n's λ-uniformly* if for each $n \in \mathbb{N}$ there is a n-dimensional subspace E_n of E such that $d(E_n, \ell_p^n) < \lambda$.
We say that E contains ℓ_p^n's uniformly if it contains ℓ_p^n's λ-uniformly for some $\lambda > 1$.

Because of Dvoretzky's Theorem every infinite dimensional Banach space contains ℓ_2^n's uniformly. A proof of this amazing theorem can be found in [15, Ch. 19]. A characterization of Banach spaces containing ℓ_p^n's for some $1 \le p < 2$ is presented in the next section.
Finally we remember, that every Banach space E and its bidual E'' have the same local structure. A proof of this can be found e.g. in [14, p. 75].

Theorem 1.9 (Principle of local reflexivity) *Let X be a Banach space and $\varepsilon > 0$. Then for every finite dimensional subspace E of X'' there exists a subspace F of X such that $d(E,F) \le 1 + \varepsilon$.*

1.4 Some notions on type and cotype

For the following definition we need the sequence of *Rademacher functions* (r_n), where the n^{th} Rademacher function is defined by $r_n(t) = \mathrm{sign}(\sin(2^n \pi t))$ for $t \in [0,1]$.

Definition 1.10 A Banach space E is of *type* p, $1 \le p \le 2$, if there exists a constant $c > 0$, such that for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in E$ we have

$$\left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|_E^2 dt \right)^{1/2} \le c \left(\sum_{i=1}^n \|x_i\|_E^p \right)^{1/p} .$$

If this inequality only holds for all finite sequences $x_1, \dots, x_n$ in the unit sphere, E is called to be of *equal-norm type* p.

For Banach spaces in which a converse inequality holds we give the following notation:

Definition 1.11 A Banach space E is of *cotype* q, $2 \le q \le \infty$, if there exists a constant $c > 0$, such that for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in E$ we have

$$\left(\sum_{i=1}^n \|x_i\|_E^q \right)^{1/q} \le c \left(\int_0^1 \left\| \sum_{i=1}^n r_i(t) x_i \right\|_E^2 dt \right)^{1/2} .$$

To cover the case $q = \infty$, the left hand side should be replaced by $max_{k \le n} \|x_k\|_E$.
If this inequality only holds for all finite sequences $x_1, \dots, x_n$ in the unit sphere, we say that E is of *equal-norm cotype* q.

Trivially, a Banach space of type p is also of type r for all $1 \le r \le p$. Analogously, a Banach space of cotype q is also of cotype r for all $q \le r \le \infty$. It is also easy to check that every Banach space has type 1 and cotype ∞. A Banach space is called *B-convex*, if it is of some type $p > 1$.
Banach spaces always have the same type or cotype as their bidual. This follows by an easy application of the principle of local reflexivity. Moreover, if E is of type p, its dual is of cotype p'. The converse is only true for B-convex spaces (cf. [15, Prop. 11.10 and Prop. 13.17]). Furthermore, a Banach space is B-convex, if and only if its dual is. This can be found in [15, Cor. 13.7].
Hilbert spaces have type and cotype 2, and these are the only ones up to isomorphy due to Kwapien's Theorem (cf. [15, Th. 12.19]). The introduced $\mathcal{L}_p$-spaces are of type $\min\{2, p\}$ and of cotype $\max\{2, p\}$ for $1 \le p < \infty$. In particular this holds for the Lebesgue spaces $L_p(\mu)$. The spaces $L_\infty(\mu)$ and $C(K)$, or more generally, $\mathcal{L}_\infty$-spaces are only of trivial type and cotype, provided that they have infinitely many dimensions.
We know from Pisier's Theorem (cf. [15, Th. 13.3] that a Banach space is B-convex, if and only if it does not contain ℓ_1^n's uniformly. But with the help of a different notion on type that we present in the following, much more can be said on Banach spaces containing ℓ_p^n's uniformly for some $1 \le p \le 2$. For this a random variable X on a measure space $(\Omega, \mathcal{A}, \mu)$ is called *standard p-stable* for some $0 < p \le 2$, if its Fourier transform is of the form

$$\int e^{itX(\omega)} \mu(d\omega) = \exp(-|t|^p/2)$$

for all $t \in \mathbb{R}$. Information on p-stable random variables can be found in [26, Ch. 5]. With the help of the above definition we can introduce another notion of type:

Definition 1.12 Let $1 \le p < 2$ and (π_i) be a sequence of independent, standard p-stable random variables on a measure space $(\Omega, \mathcal{A}, \mu)$. A Banach space E is said to be of *stable type* p, if there is a constant $c > 0$ and some $0 < r < p$ such that for all $x_1, \ldots, x_n \in E$

$$\left(\int_\Omega \left\| \sum_{i=1}^n \pi_i(\omega) x_i \right\|_E^r \mu(d\omega) \right)^{1/r} \le c \left(\sum_{i=1}^n \|x_i\|_E^p \right)^{1/p} .$$

If a Banach space is of stable type p, then it is of type p. Conversely, if it is of type p, it is also of stable type r for all $1 \le r < p$. This can be found in [26, Prop. 9.12]. We mainly consider the concept of stable type because it allows the characterization of Banach spaces containing ℓ_p^n uniformly:

Theorem 1.13 *Let $1 \le p < 2$. A Banach space contains ℓ_p^n's uniformly, if and only if it is not of stable type p.*

For a proof of this theorem we refer to [26, Th. 9.6]. The following Theorem, which can be found in [26, Cor. 9.7] is also important for some of our considerations. Roughly speaking it states that no Banach space has got a maximal stable type:

Theorem 1.14 *Let $1 \le p < 2$. If a Banach space is of stable type p it is also of stable type r for some $r > p$.*

In the sequel we also need a weaker notion on type and cotype. For this a random variable X on a probability space $(\Omega, \mathcal{A}, \mu)$ is called *standard Gaussian*, if its distribution satisfies

$$\mu(X^{-1}(B)) \;=\; \frac{1}{\sqrt{2\pi}} \int_B e^{-t^2/2} dt$$

for all Borel sets $B \subset \mathbb{R}$. Moreover, given a sequence of independent standard Gaussian variables (g_n) on $(\Omega, \mathcal{A}, \mu)$, we define the γ-summing norm of an operator $T : \ell_2^n \to E$ by

$$\pi_\gamma(T) \;:=\; \left(\int_\Omega \left\| \sum_{k=1}^n g_k(\omega)\, Te_k \right\|_E^2 \mu(d\omega) \right)^{1/2} .$$

Here $e_1, \ldots, e_n$ denotes the canonical basis of ℓ_2^n. Furthermore for $T : E \to \ell_2^n$ we define

$$\pi_\gamma^*(T) \;:=\; \sup\{ \text{ trace}(TS) \mid S : \ell_2^n \to E \text{ with } \pi_\gamma(S) \le 1 \} ,$$

which is in fact the adjoint norm of π_γ. For details on the π_γ-summing norm we refer to [15, Ch. 12]. Now we are able to give the following two definitions:

Definition 1.15 A Banach space E is of *weak type* p, $1 < p \leq 2$ if there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all operators $T : E \to \ell_2^n$:

$$\sup_{n \geq 1} n^{1-1/p} \; a_n(T) \;\leq\; c \; \pi_\gamma^*(T) \; .$$

Definition 1.16 A Banach space E is of *weak cotype* q, $2 \leq q < \infty$ if there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all operators $T : \ell_2^n \to E$:

$$\sup_{n \geq 1} n^{1/q} \; a_n(T) \;\leq\; c \; \pi_\gamma(T) \; .$$

A Banach space E is of weak type p, if and only if E is B-convex and E' is of weak cotype p' (cf. [30, Th.3.3.]). Moreover we have:

Proposition 1.17 *A Banach space E has always the same weak type or weak cotype as its bidual.*

Proof: Because of the above remark it suffices to prove the statement for cotype q. For this we first observe that $\pi_\gamma(T) = \pi_\gamma(IT)$ and $a_n(T) = c_n(T) = c_n(IT) = a_n(IT)$ for any operator $T : \ell_2^n \to E$ and every metric injection I. Now the implication from E'' to E is trivial. The converse follows by an application of the principle of local reflexivity. ◀

It is known that cotype q implies weak cotype q (cf. [30, Lem. 1.3. and p. 83]) and that conversely, weak cotype q implies cotype r for all $q < r \leq \infty$ (cf. [30, p. 93]). Analogue statements hold for the relation between type and weak type. In particular a Banach space is B-convex, if and only if it is of some weak type $p > 1$. However, weak (co)-type q does not imply (co)-type q (cf. [30, p. 93]).
Finally, in contrast with the fact that equal-norm type 2 coincides with type 2 (this is due to Pisier, see [21]), equal-norm type $p \in (1, 2)$ is the same as weak type p (cf. [30, Th. 3.4.]). Again, an analogue statement holds for the weak cotype case (cf. [30, Th. 2.3.]). For further details on weak type and cotype we also refer to [30].

1.5 Duality of entropy numbers - some results

The duality problem of entropy numbers concerns the question whether there is a relation between the entropy numbers of an arbitrary operator and its dual operator. Roughly speaking it asks, whether there are constants $a, c > 0$ such that for every operator $T : E \to F$ it holds

$$c^{-1} \; e_{ak}(T') \;\leq\; e_k(T) \;\leq\; c \; e_{\lfloor k/a \rfloor}(T') \; .$$

In a weaker form it conjectures that $(e_n(T))$ and $(e_n(T'))$ have at least similar asymptotic behaviour, for example in the sense of

$$c^{-1} \ \|(e_n(T'))\| \ \leq \ \|(e_n(T))\| \ \leq \ c \ \|(e_n(T'))\|$$

for any symmetric, i.e. permutation invariant, norm $\|.\|$ defined for sequences. Although the general case is open, there are some powerful results. We begin with the following theorem of Bourgain, Pajor, Szarek and Tomczak-Jaegermann which can be found in [2, Th. 3].

Theorem 1.18 *Let E and F be Banach spaces such that one of them is B-convex. Then for every $0 < p < \infty$ there exists a constant $c \geq 1$, such that for all compact operators $T : E \to F$ and all $n \geq 1$ we have*

$$c^{-1} \ \sup_{k \leq n} k^{1/p} \ e_k(T) \ \leq \ \sup_{k \leq n} k^{1/p} \ e_k(T') \ \leq \ c \ \sup_{k \leq n} k^{1/p} \ e_k(T) \ .$$

A little trick due to Carl (cf. [6, p. 106]) which we later use again gives us the following corollary we often need:

Corollary 1.19 *Let E and F be Banach spaces such that one of them is B-convex and $T : E \to F$ be a compact operator. Moreover let (a_n) be a regular sequence. Then we have*

$$e_n(T) \ \preceq \ a_n \qquad \textit{if and only if} \qquad e_n(T') \ \preceq \ a_n$$

and

$$e_n(T) \ \sim \ a_n \qquad \textit{if and only if} \qquad e_n(T') \ \sim \ a_n \ .$$

Proof: Let $c > 0$ such that $a_n \leq c a_{2n}$ and $a_m \leq c a_n$ for all $n \leq m$. We define $r := \frac{1}{\log_2 c}$ and remember

$$a_n \leq c^2 \ \left(\frac{m}{n}\right)^{1/r} \ a_m$$

for $n \leq m$. Hence the first equivalence is a direct consequence of Theorem 1.18 in the case of $p = r$.
We now assume $e_n(T) \sim a_n$. Then we already know $e_n(T') \leq c_1 \ a_n$. Moreover we have $a_n \leq c^2 \ m^{1/r} \ a_{m \cdot n}$ for all $m, n \in \mathbb{N}$. Hence for $p \in (0, r)$ and suitable constants $c_2, c_3 \geq 1$ we obtain:

$$\begin{aligned}
(m \cdot n)^{1/p} \ a_{m \cdot n} \ &\leq \ c_2 \ (m \cdot n)^{1/p} \ e_{m \cdot n}(T) \\
&\leq \ c_3 \ \sup_{k \leq m \cdot n} \ k^{1/p} \ e_k(T') \\
&\leq \ c_3 \ \left(\sup_{k \leq n} k^{1/p} \ e_k(T') + \sup_{n \leq k \leq m \cdot n} \ k^{1/p} \ e_k(T') \right) \\
&\leq \ c_1 \ c_3 \ c^2 \ n^{1/p} \ a_n + c_3 \ e_n(T') \ (m \cdot n)^{1/p} \\
&\leq \ c_1 \ c_3 \ c^4 \ n^{1/p} \ m^{1/r} a_{m \cdot n} + c_3 \ e_n(T') \ (m \cdot n)^{1/p}
\end{aligned}$$

by Theorem 1.18. Hence, if we choose $m \in \mathbb{N}$ with $m^{1/p-1/r} > c_1 \cdot c_3 \cdot c^4$ we get $e_n(T') \sim a_n$. The converse implication can be proven analogously. ◄

For an operator T with fixed finite rank it is also possible to compare some entropy numbers of T *directly* with some of T'. This is due to König and Milman (cf. [23, Th. 3] and [35, Cor. 8.11.]).

Theorem 1.20 *For every $a > 0$ there is a constant $c > 1$ such that for any finite rank operator $T : E \to F$ between real Banach spaces and all $n > a \cdot \operatorname{rank} T$ we have*

$$e_{\lfloor cn \rfloor}(T) \;\le\; 2e_n(T') \qquad \textit{and} \qquad e_{\lfloor cn \rfloor}(T') \;\le\; 2e_n(T) \;.$$

Using $\operatorname{rank} T = \operatorname{rank} T' = \operatorname{rank} T''$ for every finite rank operator T we immediately obtain the following corollary:

Corollary 1.21 *There is a constant $c \in \mathbb{N}$ such that for every operator $T : E \to F$ with* $\operatorname{rank} T = n$ *we have*

$$e_{cn}(T'') \;\le\; 4e_n(T) \;.$$

1.6 Local estimates of entropy numbers

In this section we introduce some so-called 'local estimates' of entropy numbers which play a fundamental role for our results. Thereby our aim is to systemize both well-known results and duality relations. For the sake of clarity we give some ad-hoc definitions. We begin with:

Definition 1.22 A Banach space E is said to be of *entropy type p*, $1 < p \le 2$, if there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all operators $T : \ell_1^n \to E$ we have

$$e_k(T) \;\le\; c \, \left(\frac{\log_2(\frac{n}{k}+1)}{k} \right)^{1-1/p} \|T\| \;, \qquad\qquad 1 \le k \le n \;.$$

One easily checks with the principle of local reflexivity that a Banach space is of entropy type p, if and only if its bidual is. Moreover, there are important connections to the concepts of type and weak type which are described in the next theorem:

Theorem 1.23 *Let E be a Banach space and $1 < p < 2$. Then the following statements are equivalent:*

i) E is of weak type p.

ii) E is of entropy type p.

iii) For every $0 < r < p'$ *there exists a constant* $c > 0$ *such that for all* $n \in \mathbb{N}$ *and all operators* $T : \ell_1^n \to E$ *it holds:*

$$\sup_{k \geq 1} k^{1/r} \; e_k(T) \; \leq \; c \; n^{1/r-(1-1/p)} \; \|T\| \; .$$

In this case we say that E is of weak entropy type p.

Moreover, in the case of $p = 2$ *the following implications hold:*

$$E \textit{ is of type 2} \quad \Rightarrow \quad ii) \quad \Rightarrow \quad iii) \quad \Rightarrow \quad i)$$

Proof: The implication $i) \to ii)$ is due to Junge and M. Defant in [22, Th. 1]. In a weaker form it originally goes back to Maurey (cf. [34] and [6, Prop. 1]). From $ii)$ to $iii)$ one gets by a simple estimation. From $iii)$ we infer that

$$e_n(T) \; \leq \; c \; n^{-(1-1/p)} \; \|T\|$$

for every operator $T : \ell_1^n \to E$ and all $n \in \mathbb{N}$. But this implies that E is of weak type p by [22, Th. 1].

In the case $p = 2$ the implication $iii) \to i)$ is due to Pajor [32]. The fact that type 2 implies entropy type 2 was proven by Maurey (cf. [34] and [6, Prop. 1]). ◄

Apart from $\lambda^{(e)}_{r,\infty}(T : \ell_1^n \to E)$ for $0 < r < p'$ we are also interested in the limiting case $\lambda^{(e)}_{p',\infty}(T : \ell_1^n \to E)$. Although the estimation is very easy we want to point out the result in a lemma:

Lemma 1.24 *Assume that* E *is of entropy type* p, $1 < p \leq 2$. *Then there exists a constant* $c > 0$ *such that for all operators* $T : \ell_1^n \to E$ *we have*

$$\sup_{k \geq 1} k^{1-1/p} \; e_k(T) \; \leq \; c \; \|T\| \; (\log_2(n+1))^{1-1/p} \; .$$

We now consider the 'dual' situation:

Definition 1.25 A Banach space E is said to be of *entropy cotype* q, $2 \leq q < \infty$, if there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all operators $T : E \to \ell_\infty^n$ we have

$$e_k(T) \; \leq \; c \; \left(\frac{\log_2(\frac{n}{k}+1)}{k} \right)^{1/q} \; \|T\| \; , \qquad\qquad 1 \leq k \leq n \; .$$

To transfer the statements of Theorem 1.23 to the case of entropy cotype we need some additional information:

Lemma 1.26 *Let* (a_n) *be a positive sequence with* $\liminf_{n \to \infty} a_n = 0$. *Moreover, let* E *be a Banach space such that for all* $n \in \mathbb{N}$ *and all operators* $T : E \to \ell_\infty^n$ *we have* $e_n(T) \leq a_n \; \|T\|$. *Then* E *must be B-convex.*

In particular, if E *is of some entropy cotype* $q < \infty$, *then* E *is B-convex.*

Proof: Suppose E is not B-convex. Then E' is not B-convex either and therefore it contains ℓ_1^n's uniformly by Pisier's Theorem. Hence without loss of generality there are subspaces $E_n \subset E'$ and isomorphisms $T_n : E_n \to \ell_1^n$ with $\|T_n\| = 1$ and $\|T_n^{-1}\| \leq 2$. Let I_n be the embedding of E_n into E' and $S_n := I_n T_n^{-1} : \ell_1^n \to E'$. Then we have $e_k(S_n') = e_k((T_n^{-1})')$ for all $k \geq 1$ since I_n' is a metric surjection. We define $R_n := (S_n')_{|E} : E \to \ell_\infty^n$. One easily checks $S_n = R_n'$ and hence $S_n' = R_n''$. Now let $c > 0$ be the constant appearing in Corollary 1.21. By Lemma 1.5 we then obtain that

$$2^{-c} \leq e_{cn}(id_{\ell_\infty^n}) = e_{cn}((T_n^{-1})'T_n') \leq e_{cn}((T_n^{-1})') = e_{cn}(R_n'') \leq 4\, e_n(R_n) \leq 8\, a_n$$

holds for all $n \geq 1$. But this contradicts the assumption on (a_n). ◀

Proposition 1.27 *For every Banach space E and $1 < p \leq 2$ the following conditions are equivalent:*

i) E is of entropy cotype p'.

ii) E' is of entropy type p.

In particular E and E'' always have the same entropy cotype.

Proof: Suppose that E is of entropy cotype p' and let $T : \ell_1^n \to E'$ be an arbitrary operator. We define $S := T'_{|E} : E \to \ell_\infty^n$. Then we have $S' = T$ and an application of Lemma 1.26 together with Theorem 1.18 gives the desired assertion.
Conversely, if E' is of entropy type p then E' is B-convex by Theorem 1.23. Again an application of Theorem 1.18 yields the assertion. ◀

Now we are able to 'dualize' Theorem 1.23:

Theorem 1.28 *Let E be a Banach space and $2 < q < \infty$. Then the following statements are equivalent:*

i) E is B-convex and of weak cotype q.

ii) E is of entropy cotype q.

iii) For every $0 < r < q$ there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all operators $T : E \to \ell_\infty^n$ holds:

$$\sup_{k \geq 1} k^{1/r}\, e_k(T) \leq c\, n^{1/r-1/q}\, \|T\| .$$

In this case we say that E is of weak entropy cotype *q.*

Moreover, in the case $q = 2$ the following implications hold:

$$E \text{ is } B\text{-convex and of cotype 2} \;\Rightarrow\; ii) \;\Rightarrow\; iii) \;\Rightarrow\; i)$$

Proof: Suppose first that E is B-convex and of weak cotype q. Then E' is of weak type q' and therefore of entropy type q'. Hence Proposition 1.27 tells us that E is of entropy cotype q.
Now suppose that $iii)$ holds. Then we see that E is $B-$convex by Lemma 1.26. Moreover, repeating the proof of Proposition 1.27 we obtain that E' is of weak entropy type q'. But then E' is of weak type q' and hence E is of weak cotype q.
In the case of $q = 2$ the proof is similar. ◀

Furthermore, we obtain analogously to Lemma 1.24:

Lemma 1.29 *Assume that E is of entropy cotype q, $2 \leq q < \infty$. Then there exists a constant $c > 0$ such that for all $T : E \to \ell_\infty^n$ we have*

$$\sup_{k \geq 1} k^{1/q} \ e_k(T) \ \leq \ c \ \|T\| \ (\log_2(n+1))^{1/q} \ .$$

Up to now we have collected a lot of local properties a Banach space may have. We summarize some of them in a diagram to avoid confusions:

$$\begin{tikzcd}
\begin{array}{c}E \text{ does not contain} \\ (\ell_p^n)\text{'s uniformly}\end{array} \arrow[d, leftrightarrow, "1 \leq p < 2"] & \begin{array}{c}E \text{ is an } \mathcal{L}_q\text{-space,} \\ 1 \leq q < \infty\end{array} \arrow[d, "p = \min\{2,q\}"] & & \\
\begin{array}{c}E \text{ is of stable} \\ \text{type } p\end{array} \arrow[r, "1 \leq p < 2"] \arrow[d, "1 \leq p < 2"] & \begin{array}{c}E \text{ is of} \\ \text{type } p\end{array} \arrow[r, "1 < p \leq 2"] \arrow[d, "1 < p \leq 2"] & \begin{array}{c}E \text{ is of weak} \\ \text{type } p\end{array} \arrow[r, "1 < p \leq 2"] \arrow[d, shift right, "1 < p < 2"'] & \begin{array}{c}E \text{ is of stable} \\ \text{type } p - \varepsilon\end{array} \\
\begin{array}{c}E \text{ is of some} \\ \text{stable type } p + \varepsilon\end{array} & \begin{array}{c}E \text{ is of entropy} \\ \text{type } p\end{array} \arrow[r, shift left, "1 < p \leq 2"] & \begin{array}{c}E \text{ is of weak} \\ \text{entropy type } p\end{array} \arrow[l, shift left, "1 < p < 2"] \arrow[u, shift right, "1 < p \leq 2"'] &
\end{tikzcd}$$

Note that we have dropped out the duality relations between the several notions of type and cotype. However, the implications shown in the diagram are the most useful ones for our later work.
Finally we point out that in some cases there are similar local estimates to the discussed ones for the Gelfand and Kolmogorov numbers:

Remark 1.30 Let $T : \ell_1^n \to E$ be an arbitrary operator and $\pi_2(T)$ its 2-absolutely summing norm (cf. [15]). Then Carl and Pajor proved in [11, Th. 2.2.] that there is a universal constant $c > 0$ such that

$$c_k(T) \ \leq \ c \ \left(\frac{\log_2(\frac{n}{k} + 1)}{k} \right)^{1/2} \ \pi_2(T) \ , \qquad 1 \leq k \leq n \ .$$

Note that for Hilbert spaces E the 2-absolutely summing norm of T can be replaced by the operator norm $\|T\|$ if we enlarge the constant c because of Grothendiecks Theorem. (The Hilbert space case is what Carl and Pajor actually proved.) Hence for Hilbert spaces analogue estimates to those of the entropy type definition hold with $q = 2$. Moreover, by duality the same estimates hold for the Kolmogorov numbers $d_n(T)$ of operators $T : E \to \ell_\infty^n$ starting in a Hilbert space E.
Now let us assume that E is an $\mathcal{L}_{p,\lambda}$-space for some $1 < \lambda, p < \infty$ and $T : \ell_1^n \to E$. Since π_2 is an injective norm and

$$d(\mathrm{im}\, T, \ell_2^{\mathrm{rank}\ T}) \ \leq \ \lambda\ n^{|1/2-1/p|}$$

by a theorem of Lewis (cf. [27] and [40, III.B. Cor. 9]), one easily checks that

$$\pi_2(T) \ \leq \ c\ n^{|1/2-1/p|} \ \|T\|$$

for some constant $c > 0$ only depending on E. Hence for all $0 < r < 2$ there is a constant $c_r > 0$ such that

$$\sup_{k\geq 1} k^{1/r}\ c_k(T) \ \leq \ c_r\ n^{1/r-\min\{1/p,1/p'\}}\ \|T\| \ . \tag{1.1}$$

Analogously, if $T : E \to \ell_\infty^n$ is an operator starting in an $\mathcal{L}_p$-space and $0 < r < 2$, we obtain a constant $c_r > 0$ such that

$$\sup_{k\geq 1} k^{1/r}\ d_k(T) \ \leq \ c_r\ n^{1/r-\min\{1/p,1/p'\}}\ \|T\| \ . \tag{1.2}$$

We will discuss some consequences of the above estimates at the end of Chapter 3.

Chapter 2

Decomposition of Operators

In this chapter our aim is to show how a 1-Hölder-continuous operator $T : E \to C(K)$ can be decomposed into finite sums of type

$$T = \sum_{i=0}^{n} T_i + S \ ,$$

where the operators $T_i : E \to C(K)$ essentially map into some spaces $\ell_\infty^{\alpha_i}$. Thereby we also have to control both the norms of the operators $T_o, \ldots, T_n, S$ and the dimensions α_i in terms of the entropy numbers of the underlying compact metric space (K, d). Our construction is inspired by the earlier work of Carl, Heinrich and Kühn (cf. [9, Lem. 1 and 2] and [12, Lem. 5.10.1. and 5.10.2.]).
We begin with a simple lemma which describes how a 1-Hölder-continuous operator can be approximated with the help of a partition of unity:

Lemma 2.1 *Let (K, d) be a compact metric space and $T : E \to C(K)$ be a 1-Hölder-continuous operator. Furthermore, let $\varphi_1, \ldots, \varphi_n \in C(K)$ be a partition of unity and $t_1, \ldots, t_n \in K$ such that $\varphi_i(t_j) = \delta_{ij}$. Then for the operator*

$$\begin{aligned} A : E &\to C(K) \\ x &\mapsto \sum_{i=1}^{n} Tx(t_i) \ \varphi_i \end{aligned}$$

we have:

$$\begin{aligned} \|A\| &\leq \|T\| \\ \text{im } A &\subset \text{span } \{\varphi_1, \ldots, \varphi_n\} \stackrel{1}{=} \ell_\infty^n \\ \|T - A\| &\leq 2 \ \sup_{i \leq n} \varepsilon_1(\text{supp } \varphi_i) \ \|T\|_1 \end{aligned}$$

Proof: To show $\|A\| \leq \|T\|$, we just observe that for $x \in E$ and $t \in K$:

$$|Ax(t)| = \left|\sum_{i=1}^{n} Tx(t_i)\varphi_i(t)\right| \leq \sum_{i=1}^{n} |Tx(t_i)|\varphi_i(t) \leq \sum_{i=1}^{n} \|Tx\|\,\varphi_i(t) = \|Tx\|\,.$$

Now let us define $\varepsilon := \sup_{i \leq n} \varepsilon_1(\text{supp } \varphi_i)$. Then for $t \in \text{supp } \varphi_i$ and $x \in B_E$ we get

$$|Tx(t) - Tx(t_i)|\ \varphi_i(t) \leq 2\ \varepsilon\ \|T\|_1\ \varphi_i(t)\ ,$$

because T is 1-Hölder-continuous. Since the last inequality is trivial for $t \notin \text{supp } \varphi_i$ we obtain

$$\begin{aligned} |Tx(t) - Ax(t)| &= \left|\sum_{i=1}^{n} (Tx(t) - Tx(t_i))\ \varphi_i(t)\right| \\ &\leq \sum_{i=1}^{n} |Tx(t) - Tx(t_i)|\ \varphi_i(t) \\ &\leq 2\ \varepsilon\ \|T\|_1 \sum_{i=1}^{n} \varphi_i(t) = 2\ \varepsilon\ \|T\|_1\ . \end{aligned}$$

Therefore we have $\|T - A\| \leq 2\ \varepsilon\ \|T\|_1$. It remains to prove that span $\{\varphi_1, \ldots, \varphi_n\}$ is isometrically isomorphic to ℓ_∞^n. The φ_i's are linearly independent since $\varphi_i(t_j) = \delta_{ij}$. Therefore the linear map

$$\begin{aligned} I : \text{span } \{\varphi_1, \ldots, \varphi_n\} &\to \ell_\infty^n \\ \sum_{i=1}^{n} \lambda_i \varphi_i &\mapsto (\lambda_1, \ldots, \lambda_n) \end{aligned}$$

is well defined and bijective. Furthermore for $\varphi = \sum_{i=1}^{n} \lambda_i \varphi_i$ we have

$$\|\varphi\| = \sup_{t \in K} \left|\sum_{i=1}^{n} \lambda_i \varphi_i(t)\right| \leq \sup_{t \in K} \sum_{i=1}^{n} |\lambda_i| \varphi_i(t) \leq \|(\lambda_1, \ldots, \lambda_n)\|_{\ell_\infty^n}$$

and

$$\|\varphi\| = \sup_{t \in K} \left|\sum_{i=1}^{n} \lambda_i \varphi_i(t)\right| \geq \left|\sum_{i=1}^{n} \lambda_i \varphi_i(t_j)\right| = |\lambda_j|\ . \blacktriangleleft$$

Having learned how we can approximate 1-Hölder-continuous operators, our further work can be described as follows:
First we construct a finite sequence of partitions of unity $P_o, \ldots, P_n$ such that for each of them we can control its cardinality and the diameter of the supports of its functions. Then we apply the above lemma in order to build corresponding 'approximation' operators $A_o, \ldots, A_n$, and finally we define $T_i := A_i - A_{i-1}$.
For the construction of $P_o, \ldots, P_n$ we will use some sort of backward induction. For clarity's sake we first show how the induction step works:

Lemma 2.2 *Let (K,d) be a compact metric space, $(\varphi_i) \subset C(K)$ be a partition of unity, and $t_i \in K$ such that $\varphi_i(t_j) = \delta_{ij}$. Furthermore, let $M \in \mathbb{N}$ and $\delta > 0$ such that $\varepsilon_M(K) < \delta$. Then there exists a partition of unity (ψ_i) of at most M functions and $s_i \in K$ such that:*

$$\begin{aligned} \psi_i(s_j) &= \delta_{ij} \\ \varepsilon_1(\text{supp } \psi_i) &\leq \delta + \sup_j \varepsilon_1(\text{supp } \varphi_j) \\ \text{span } (\psi_i) &\subset \text{span } (\varphi_i) \end{aligned}$$

Proof: Let $\varepsilon := \sup_j \varepsilon_1(\text{supp } \varphi_j)$. Since K is compact, there are elements $y_i \in K$ such that supp $\varphi_i \subset B(y_i, \varepsilon)$. Furthermore, $\varepsilon_M(K) < \delta$ implies the existence of a δ-net $\{z_1, \ldots, z_m\} \subset K$ with $m \leq M$. Now let $A_1, \ldots, A_m$ be a partition of K with $A_i \subset B(z_i, \delta)$. Then for $1 \leq i \leq m$ we define

$$\psi_i := \sum_{y_j \in A_i} \varphi_j$$

if there exists an index j with $y_j \in A_i$. Otherwise we omit the index i. Therefore, (ψ_i) is a partition of unity of at most M functions and span $(\psi_i) \subset$ span (φ_i). Moreover, for $t \in$ supp ψ_i there exists an $y_j \in A_i$ such that $t \in$ supp $\varphi_j \subset B(y_j, \varepsilon)$. Hence $d(t, y_j) \leq \varepsilon$. On the other hand, $y_j \in A_i \subset B(z_i, \delta)$ implies $d(y_j, z_i) \leq \delta$. Therefore, we get $d(t, z_i) \leq \delta + \varepsilon$ and hence

$$\varepsilon_1(\text{supp } \psi_i) \leq \delta + \varepsilon \ .$$

Finally, let $1 \leq i \leq m$ such that there exists an index j with $y_j \in A_i$. Define $s_i := t_j$. Then for $k \leq m$ we obtain

$$\psi_k(s_i) = \sum_{y_l \in A_k} \varphi_l(s_i) = \sum_{y_l \in A_k} \varphi_l(t_j) = \sum_{y_l \in A_k} \delta_{l,j} = \delta_{i,k}$$

since $y_j \in A_k$ if and only if $i = k$. ◀

Iterating the procedure of Lemma 2.2, we obtain a finite sequence of partitions of unity with properties we are able to control by some entropy numbers of (K,d):

Lemma 2.3 *Let (K,d) be a compact metric space and $n \geq 1$ be an integer. Moreover, let $\alpha_o, a_1, \ldots, \alpha_n \in \mathbb{N}$ and $\beta_{-1}, \beta_o, \beta_1, \ldots, \beta_n > 0$ be finite sequences such that*

$$\begin{aligned} \varepsilon_{\alpha_i}(K) &< \beta_i \\ 2\beta_i &\leq \beta_{i-1} \end{aligned}$$

for all $0 \leq i \leq n$. Then there exist partitions of unity $P_o, \ldots, P_n \subset C(K)$, $P_k = (\varphi_{k,i})_i$ and elements $t_{k,i} \in K$ such that for all $0 \leq k \leq n$ we have:

$$\begin{aligned} \text{card } P_k &\leq \alpha_k \\ \varepsilon_1(\text{supp } \varphi_{k,i}) &\leq \beta_{k-1} \\ \varphi_{k,i}(t_{k,j}) &= \delta_{ij} \\ \text{span } P_k &\subset \text{span } P_{k+1} \ . \end{aligned}$$

Proof: Let $k = n$. Since $\varepsilon_{\alpha_n}(K) < \beta_n$ there exists a minimal β_n-net Γ consisting of $a_n \leq \alpha_n$ elements $x_1, \ldots, x_{a_n}$. Let $P_n = (\varphi_{n,i})_{i \leq a_n}$ be a partition of unity subordinate to this open covering. Then

$$\varepsilon_1(\text{supp }\varphi_{n,i}) \;\leq\; \varepsilon_1(B(x_i, \beta_n)) \;=\; \beta_n \;\leq\; \beta_{n-1}$$

and since Γ is minimal, we can find elements $t_{n,j} \in B(x_j, \beta_n)$ such that $t_{n,j} \notin B(x_i, \beta_n)$ for $i \neq j$. Hence $\varphi_{n,i}(t_{n,j}) = \delta_{ij}$.
Now we assume that we have already constructed P_{k+1} according to the assertion. Then by Lemma 2.2 we get a partition of unity $P_k := (\psi_i)$ of at most α_k functions and elements $(s_i)_i$ such that

$$\begin{aligned}
\varepsilon_1(\text{supp }\psi_i) &\leq \beta_k + \sup_j \varepsilon_1(\text{supp }\varphi_{k+1,j}) \;\leq\; 2\beta_k \;\leq\; \beta_{k-1} \\
\psi_i(s_j) &= \delta_{ij} \\
\text{span } P_k &\subset \text{span } P_{k+1}\;.
\end{aligned}$$

Therefore, we define $\varphi_{k,i} := \psi_i$ and $t_{k,i} := s_i$. ◀

Now we combine the previous lemma with Lemma 2.1 and obtain a decomposition of 1-Hölder-continuous operators $T : E \to C(K)$ announced at the beginning:

Lemma 2.4 *Let (K, d) be a compact metric space and $n \geq 1$ be an integer. Moreover, let $\alpha_o, a_1, \ldots, \alpha_n \in \mathbb{N}$ and $\beta_{-1}, \beta_o, \beta_1, \ldots, \beta_n > 0$ be finite sequences, such that*

$$\begin{aligned}
\varepsilon_{\alpha_i}(K) &< \beta_i \\
2\beta_i &\leq \beta_{i-1}
\end{aligned}$$

for all $0 \leq i \leq n$. Furthermore, let $T : E \to C(K)$ be a 1-Hölder-continuous operator. Then there exists a decomposition

$$T = \sum_{i=0}^{n} T_i + S$$

by operators $T_i : E \to C(K)$ and $S : E \to C(K)$ such that

$$\begin{aligned}
\|T_i\| &\leq 4\beta_{i-2}\;\|T\|_1 && \text{for } i = 1, \ldots, n \\
\|T_o\| &\leq \|T\| \\
\|S\| &\leq 2\beta_{n-1}\;\|T\|_1 \\
\text{im } T_i &\overset{1}{\hookrightarrow} \ell_\infty^{\alpha_i} && \text{for } i = 0, \ldots, n.
\end{aligned}$$

Moreover, T_o is of the form $x \mapsto \sum_i Tx(s_i)\psi_i(.)$, where $(\psi_i)_i \subset C(K)$ is a partition of unity of at most α_o functions with $\psi_i(s_j) = \delta_{i,j}$ and $\varepsilon_1(\text{supp }\psi_i) \leq \beta_{-1}$.

Proof: By Lemma 2.3 we get partitions of unity $P_0, \ldots, P_n$. Therefore, by Lemma 2.1 we can construct operators $A_k : E \to C(K)$ with

$$\begin{aligned} \|A_k\| &\leq \|T\| \\ \|T - A_k\| &\leq 2\ \beta_{k-1}\ \|T\|_1 \\ \text{im } A_k &\subset \text{span } P_k \overset{1}{\hookrightarrow} \ell_\infty^{\alpha_k}\ . \end{aligned}$$

Now we define $T_o := A_o$, $T_i := A_i - A_{i-1}$ for $i = 1, \ldots, n$ and $S := T - A_n$. Clearly, we have $T = \sum_{i=0}^n T_i + S$ and $\|S\| \leq 2\beta_{n-1}\|T\|_1$. Furthermore,

$$\|T_i\| \ \leq \ \|T - A_i\| + \|T - A_{i-1}\| \ \leq \ 2\ \|T\|_1\ (\beta_{i-1} + \beta_{i-2}) \ \leq \ 4\beta_{i-2}\ \|T\|_1$$

holds. Since im $A_{k-1} \subset$ span P_k, we finally obtain im $T_k \subset$ span P_k. ◀

Chapter 3

Entropy of $C(K)$-valued operators

Given a 1-Hölder-continuous operator $T : E \to C(K)$, one can ask how the entropy numbers of T are influenced by those of the underlying compact metric space (K, d). If we use

$$a_{k+1}(T) \ \leq \ \omega(T, \varepsilon_k(K)) \ \leq \ \|T\|_1 \ \varepsilon_k(K)$$

which can be found in [12, Th. 5.6.1.], Theorem 1.4 tells us

$$\sup_{k \leq n} \ k^{1/p} \ e_k(T) \ \leq \ 2^{1/p} \ c_p \ c_K \ \|T\|_1 \ \sup_{k \leq n} \ k^{1/p} \ \varepsilon_k(K) \ , \tag{3.1}$$

where c_p is the constant of Theorem 1.4 and $c_K := \frac{1}{\min\{1, \varepsilon_1(K)\}}$. Hence if (a_n) is a regular sequence one easily checks that $\varepsilon_n(K) \preceq a_n$ implies

$$e_n(T) \ \preceq \ a_n \ .$$

However, as pointed out in the introduction, Carl, Heinrich and Kühn proved (cf. [9, Th.1], [12, Th. 5.10.1] or [10, Th. 2.3]) that if E is a Hilbert space or, more generally, if E' is of type q, then $\varepsilon_n(K) \preceq n^{-1/p}$ implies

$$e_n(T) \ \preceq \ n^{-(1-1/q)-1/p} \ .$$

Hence inequality (3.1) is not optimal in this case. Actually, it yields asymptotically optimal results in some particular sense, if and only if E is not B-convex as we will see in Chapter 5.

In this chapter we prove some variants of the above inequality (3.1) which do involve the local structure of E in terms of (weak) entropy cotype. It turns out that we have to distinguish three major types of inequalities. All of them yield asymptotically optimal results in one of the following cases:

- $\varepsilon_n(K) \ \preceq \ (\log(n+1))^{-1/p} \ f(\log(\log(n+1)+1))$,

  where p is "large". This type of decay we call *slow logarithmic.*

- $\varepsilon_n(K) \preceq (\log(n+1))^{-1/p} f(\log(\log(n+1)+1))$,

  where p is "small". This type of decay we call *fast logarithmic.*

- $\varepsilon_n(K) \preceq n^{-1/p} f(\log(n+1))$,

  where $0 < p < \infty$.

In all cases f denotes a σ-controlled function.

We will establish the announced inequalities in sections 3.1, 3.2 and 3.3. Thereafter in the forth section we show how one can estimate $(e_n(T))$ with the help of these inequalities, if one knows that $(\varepsilon_n(K))$ decreases in one of the above senses. In the last section we discuss some generalizations and give historical remarks.
Throughout this chapter we restrict ourselves to $1-$Hölder-continuous operators, since it is easy to derive similar results for $\alpha-$Hölder-continuous operators by equipping (K, d) with the new metric d^α (cf. [8]).
Moreover, all results of this section can be applied to 1-Hölder-continuous operators $T : E \to \ell_\infty(A)$, where A is a precompact metric space, since such operators factor canonically through $C(\tilde{A})$, where $\tilde{A}$ denotes the completion of A.

3.1 The case of slow logarithmic decay

As pointed out above we begin with an estimate which covers slow logarithmic decay of $(\varepsilon_n(K))$. Although one might think that this case is rather artificial since such a decay never occurs for $K \subset \mathbb{R}^n$, it turns out that the following theorem - together with an easy generalization - have impressive consequences we shall discuss in sections 4.1 and 5.3:

Theorem 3.1 *Let E be a Banach space of entropy cotype q, $2 \le q < \infty$. Then for all $p \in (q, \infty)$ there exists a constant $c \ge 1$, such that for all compact metric spaces (K, d) and all 1-Hölder-continuous operators $T : E \to C(K)$ we have*

$$\sup_{k \le n} k^{1/p} e_k(T) \le c\, c_K \|T\|_1 \sup_{k \le n} k^{1/p} e_k(K) .$$

Roughly speaking, Theorem 3.1 states that for B-convex Banach spaces and "large" p one can replace the entropy numbers $\varepsilon_n(K)$ in inequality (3.1) by the *dyadic* entropy numbers $e_n(K)$. Note that in general the latter ones decrease much faster then the former ones.
Corollary 3.5 will show how the above inequality can be used to obtain estimates on $(e_n(T))$, while we will see by Proposition 5.8 that these estimates cannot be asymptotically improved in general. Moreover, we will prove in Proposition 5.3 that Banach spaces for which the inequality of the above theorem holds for some $p \in [2, \infty)$ must be of entropy cotype $p + \varepsilon$ for all $\varepsilon > 0$.

Proof of Theorem 3.1: We first assume that $\varepsilon_1(K) \geq 1$. Then for $n = 1$ the assertion is trivial. Therefore let us additionally assume $n \geq 2$. For fixed $p \in (q, \infty)$ we define:

$$C := \sup_{k \leq n} k^{1/p} e_k(K)$$

and $r := \lfloor \frac{1}{p} \log_2(n-1) \rfloor$. To apply Lemma 2.4, we use the finite sequences

$$\begin{array}{llll} \alpha_i & := \lfloor \exp_2 2^{ip} \rfloor & \text{for} & 0 \leq i \leq r \\ \beta_i & := C \cdot 2^{-i+1/p} & \text{for} & -2 \leq i \leq r \ . \end{array}$$

Since $\lfloor \log_2 \alpha_i \rfloor + 1 \leq n$ one easily verifies $\varepsilon_{\alpha_i}(K) < \beta_i$ for $1 \leq i \leq r$. Additionally $\varepsilon_{\alpha_o}(K) = e_2(K) \leq C \cdot 2^{-1/p} < \beta_o$ and $2\beta_i \leq \beta_{i-1}$ hold. Hence we can find a decomposition

$$T = \sum_{i=0}^{r} T_i + S$$

according to Lemma 2.4. Thus, for $s := \frac{q}{1+q}$ and a suitable constant $c_1 \geq 1$ we obtain

$$\begin{aligned} n^{1/q}\ e_n(T) &\leq n^{1/q}\ e_n(\sum_{i=0}^{r} T_i) \ + \ n^{1/q}\ \|S\| \\ &\leq \sup_{j\geq 1} j^{1/q}\ e_j(\sum_{i=0}^{r} T_i) \ + \ 2\ n^{1/q}\ \beta_{r-1}\ \|T\|_1 \\ &\leq c_1 \left(\sum_{i=0}^{r} \left(\lambda_{q,\infty}^{(e)}(T_i) \right)^s \right)^{1/s} + \ C\ 2^{2+1/p+1/q}\ (n-1)^{1/q}\ 2^{-r}\ \|T\|_1 \ . \end{aligned}$$

Since $4 \cdot \beta_{-2} = 4 \cdot 2^{2+1/p} \cdot C \geq 1$ we observe $\|T_o\| \leq \|T\| \leq \|T\|_1 \leq 4\beta_{-2}\|T\|_1$. Hence for $0 \leq i \leq r$ we may estimate

$$\begin{aligned} \lambda_{q,\infty}^{(e)}(T_i) &\leq 2\ c_2\ \|T_i\|\ (\log_2(\alpha_i+1))^{1/q} \\ &\leq 2^4\ c_2\ \beta_{i-2}\ \|T\|_1\ (\log_2 \alpha_i)^{1/q} \\ &= 2^{6+1/p}\ c_2\ C\ \|T\|_1\ 2^{i(\frac{p}{q}-1)}\ , \end{aligned}$$

where $c_2 > 0$ is the constant appearing in Lemma 1.29. Thus with $c_3 := 2^{6+1/p}\ c_1\ c_2$ and $c_4 := \frac{2^{\frac{p}{q}-1}}{(2^{s(\frac{p}{q}-1)}-1)^{1/s}}$ we receive

$$\begin{aligned} c_1 \left(\sum_{i=0}^{r} (\sup_{j\geq 1} j^{1/q}\ e_j(T_i))^s \right)^{1/s} &\leq c_3\ C\ \|T\|_1 \left(\sum_{i=0}^{r} \left(2^{i(\frac{p}{q}-1)} \right)^s \right)^{1/s} \\ &\leq c_3\ c_4\ C\ \|T\|_1\ 2^{r(\frac{p}{q}-1)}\ . \end{aligned}$$

Hence for $c_5 := c_3\ c_4 + 2^{3+1/p+1/q}$ we obtain:

$$\begin{aligned}
& n^{1/q}\ e_n(T) \\
\leq\ & c_3\ c_4\ C\ \|T\|_1\ 2^{r(\frac{p}{q}-1)} +\ C\ 2^{2+1/p+1/q}\ (n-1)^{1/q}\ 2^{-r}\ \|T\|_1 \\
\leq\ & \|T\|_1\ C\ \left(c_3\ c_4\ 2^{\frac{1}{p}\log_2(n-1)\ (\frac{p}{q}-1)}\ +\ 2^{2+1/p+1/q}\ (n-1)^{1/q}\ 2^{-\frac{1}{p}\log_2(n-1)+1}\right) \\
=\ & c_5\ \|T\|_1\ C\ (n-1)^{1/q-1/p}\ ,
\end{aligned}$$

i.e. the assertion is proven in the case $\varepsilon_1(K) \geq 1$.
Now let us assume that $\varepsilon_1(K) < 1$. Then $\tilde{d}(s,t) := \varepsilon_1(K)^{-1}\ d(s,t)$ defines a new, equivalent metric on K with $\varepsilon_n^{(\tilde{d})}(K) = \varepsilon_1(K)^{-1}\ \varepsilon_n^{(d)}(K)$ and

$$|T|_1^{(\tilde{d})} = \varepsilon_1(K)\ |T|_1^{(d)} \leq |T|_1^{(d)}\ .$$

Hence we obtain $\varepsilon_1^{(\tilde{d})}(K) = 1$ and $\|T\|_1^{(\tilde{d})} \leq \|T\|_1^{(d)}$. Using the first case we finally receive:

$$\begin{aligned}
n^{1/p} e_n(T) &\leq\ c\ \|T\|_1^{(\tilde{d})} \sup_{k\leq n} k^{1/p} e_k^{(\tilde{d})}(K) \\
&\leq\ c\ \varepsilon_1^{(d)}(K)^{-1}\ \|T\|_1^{(d)} \sup_{k\leq n} k^{1/p} e_k^{(d)}(K) \quad .◀
\end{aligned}$$

3.2 The case of fast logarithmic decay

We shall now establish an inequality which corresponds with the case of $(\varepsilon_n(K))$ decreasing essentially like $(\log(n+1))^{-1/p}$ with "small" p. It turns out that this estimate has got the most technical proof:

Theorem 3.2 *Let E be a Banach space of entropy cotype q, $2 \leq q < \infty$. Then for all $p \in (0,q)$ and all σ-controlled functions $f : [0,\infty) \to (0,\infty)$ with $0 < \sigma < \frac{1}{p} - \frac{1}{q}$ there exists a constant $c \geq 1$, such that for all compact metric spaces (K,d) and all 1-Hölder-continuous operators $T : E \to C(K)$ we have*

$$\sup_{k\leq n} k^{\frac{1}{q}}\ (\log_2(k+1))^{\frac{1}{p}-\frac{1}{q}}\ f(\log_2(k+1))\ e_k(T)\ \leq\ c \cdot c_K\ \|T\|_1 \sup_{k\leq a_n} k^{\frac{1}{p}}\ f(k)\ e_k(K)\ ,$$

where $a_n := n^{\frac{p}{q(1-\sigma p)}} \log_2(n+1)$.

Corollary 3.5 will show how the above inequality can be used to obtain estimates on $(e_n(T))$ whenever the sequence $(e_n(K))$ decreases essentially like $n^{-1/p}$ for some $p \in (0,q)$. Moreover, in Proposition 5.8 these estimates turn out to be asymptotically optimal for some 1-Hölder-continuous operator $T : E \to C(K)$, if E is not of any

better entropy cotype than q. Conversely, we will prove in Proposition 5.4 that Banach spaces for which the inequality of the above theorem holds for some $q \in [2,\infty)$ and $p \in (0,\infty)$ must be of entropy cotype $q+\varepsilon$ for all $\varepsilon > 0$.
Before we prove Theorem 3.2, we need an additional lemma, which is rather technical:

Lemma 3.3 *Assume that E is a Banach space of entropy cotype $q \in [2,\infty)$. Let $p \in (0,q)$ and $f : [0,\infty) \to (0,\infty)$ be a σ-controlled function with $0 < \sigma < \frac{1}{p} - \frac{1}{q}$. Furthermore, let K be a compact metric space with $\varepsilon_1(K) \geq 1$ and $T : E \to C(K)$ be a 1-Hölder-continuous operator. Moreover, let $n \geq 2$ and $\varphi_1, \ldots, \varphi_m \in C(K)$ be a partition of unity with $m \leq n$ and $\varphi_i(t_j) = \delta_{i,j}$ for suitable elements $t_j \in K$. Then for the operator*

$$\begin{array}{rcl} A : E & \to & C(K) \\ x & \mapsto & \displaystyle\sum_{i=1}^{m} Tx(t_i)\ \varphi_i \end{array}$$

we have

$$e_n(A) \leq c\ \|T\|_1\ n^{-1/q} \left(\sup_{i \leq m} \varepsilon_1(\text{supp } \varphi_i) + \frac{\sup_{i\leq n}(\log_2(i+1))^{1/p}\ f(\log_2 i)\ \varepsilon_i(K)}{(\log_2 n)^{1/p}\ f(\log_2 n)} \right) ,$$

where $c =: c_{p,q,f}(E)$ is a constant only depending on p, q, f and E.

Proof of Lemma 3.3: Let $c_1 \in \mathbb{N}$ with $c_1 > 6$ and $2^{4+1/p+\sigma}\ c_1^{1/q}\ 2^{-\frac{c_1}{6}} \leq 1/2$. Moreover let $c_q(E)$ be a constant such that the entropy cotype q inequality for E holds for. We define

$$\begin{array}{rcll} C_n & := & \displaystyle\sup_{i\leq n}\ (\log_2(i+1))^{1/p}\ f(\log_2 i)\ \varepsilon_i(K) & \qquad \text{for } n \geq 2 \\ c_2 & := & c_1^2 & \\ c_3 & := & c_2^{1/q} \cdot \max\left\{1, \dfrac{f(1)\cdot(\log_2 c_2)^{1/p+\sigma}}{f(0)}\right\} & \\ c_{p,q,f}(E) & := & \max\{c_3, 3^{1/q}\ 2^{4+1/p+1/q+\sigma}\ c_q(E)\} & \end{array}$$

We proceed by induction on n. For $2 \leq n \leq c_2$ and A according to the assumption we have

$$\begin{array}{rcl} 1 & \leq & (\log_2 c_2)^{1/p} \displaystyle\max_{2\leq i\leq c_2} f(\log_2 i)\ \ (\log_2 n)^{-1/p}(f(\log_2 n))^{-1} \\ & \leq & (\log_2 c_2)^{1/p} \displaystyle\max_{2\leq i\leq c_2} (\log_2 i)^{\sigma} f(1)\ \ (\log_2 n)^{-1/p}(f(\log_2 n))^{-1} \\ & \leq & \displaystyle\sup_{i\leq m} \varepsilon_1(\text{supp } \varphi_i) + (\log_2 c_2)^{1/p+\sigma}\ f(1)\ (\log_2 n)^{-1/p}(f(\log_2 n))^{-1}(f(0))^{-1}\ C_n \\ & \leq & \dfrac{c_3}{c_2^{1/q}} \left(\displaystyle\sup_{i\leq m} \varepsilon_1(\text{supp } \varphi_i) + (\log_2 n)^{-1/p}\ (f(\log_2 n))^{-1}\ C_n \right) , \end{array}$$

since $C_n \geq f(0)$. Therefore we receive

$$e_n(A) \;\leq\; c_{p,q,f}(E)\; \|T\|_1\; n^{-1/q} \left(\sup_{i \leq m} \varepsilon_1(\text{supp } \varphi_i) + \frac{C_n}{(\log_2 n)^{1/p}\; f(\log_2 n)} \right) \; .$$

Now let $n > c_2$ and A be according to the assumption. We define $M := \left\lfloor \frac{n}{c_1} \right\rfloor + 1$ and $\varepsilon := \sup_{i \leq m} \varepsilon_1(\text{supp } \varphi_i)$. Since $c_1 < \sqrt{n}$, we have $\frac{1}{2} \log_2 n \;\leq\; \log_2 M \;<\; \log_2 n$. Hence we obtain

$$\begin{aligned} (\log_2 M)^{-1/p}\; (f(\log_2 M))^{-1} \;&\leq\; 2^{1/p}\; (\log_2 n)^{-1/p} \; \left(\frac{\log_2 n}{\log_2 M} \right)^{\sigma} (f(\log_2 n))^{-1} \\ &\leq\; 2^{1/p+\sigma}\; (\log_2 n)^{-1/p}\; (f(\log_2 n))^{-1} \; . \end{aligned} \tag{3.2}$$

We let $\delta := C_n\; (\log_2 M)^{-1/p}\; (f(\log_2 M))^{-1}$. Since $M < n$ we get

$$\varepsilon_M(K) \;\leq\; C_n\; (\log_2(M+1))^{-1/p}\; (f(\log_2 M))^{-1} \;<\; \delta \; .$$

Thus by Lemma 2.2 there exists a partition of unity $(\psi_i) \subset C(K)$ of $k \leq M$ functions and elements $s_i \in K$ such that

$$\begin{aligned} \psi_i(s_j) \;&=\; \delta_{ij} \\ \varepsilon_1(\text{supp } \psi_i) \;&\leq\; \delta + \varepsilon \\ \text{span } (\psi_i) \;&\subset\; \text{span } (\varphi_i) \end{aligned}$$

Now we define the operators

$$\begin{aligned} B : E \;&\rightarrow\; C(K) \\ x \;&\mapsto\; \sum_{i=1}^{k} Tx(s_i)\; \psi_i \end{aligned}$$

and $S := A - B$. Then for $r := \left\lfloor \frac{n}{2} \right\rfloor$ we get

$$e_n(A) \;\leq\; e_r(B) + e_r(S) \; .$$

First we estimate $e_r(B)$. Since $M < n$, our induction hypothesis tells us that the assertion is already true for M and the operator B. Hence we may conclude

$$\begin{aligned} e_M(B) \;&\leq\; c_{p,q,f}(E)\; \|T\|_1\; M^{-1/q} \; \left((\delta \;+\; \varepsilon) + \frac{C_M}{(\log_2 M)^{1/p}\; f(\log_2 M)} \right) \\ &\leq\; c_{p,q,f}(E)\; \|T\|_1\; \left(\frac{n}{c_1} \right)^{-1/q} \; \left(\varepsilon + \frac{2^{1+1/p+\sigma}\; C_n}{(\log_2 n)^{1/p}\; f(\log_2 n)} \right) \\ &\leq\; c_{p,q,f}(E)\; 2^{1+1/p+\sigma}\; c_1^{1/q}\; \|T\|_1\; n^{-1/q} \; \left(\varepsilon + \frac{C_n}{(\log_2 n)^{1/p}\; f(\log_2 n)} \right) \end{aligned}$$

by inequality (3.2). Furthermore, $6 < c_1 < n$ implies

$$\frac{r}{M} = \frac{\lfloor \frac{n}{2} \rfloor}{\left\lfloor \frac{n}{c_1} \right\rfloor + 1} \geq \frac{\frac{n}{2} - 1}{\frac{n}{c_1} + 1} = c_1 \cdot \frac{\frac{1}{2} - \frac{1}{n}}{1 + \frac{c_1}{n}} \geq c_1 \cdot \frac{\frac{1}{2} - \frac{1}{6}}{2} = \frac{c_1}{6} \ .$$

Thus with [12, Lemma 5.10.3.] we obtain :

$$\begin{aligned} e_r(B) &\leq 8 \cdot 2^{-r/M} \ e_M(B) \\ &\leq 8 \cdot 2^{-c_1/6} \ c_{p,q,f}(E) \ 2^{1+1/p+\sigma} \ c_1^{1/q} \ \|T\|_1 \ n^{-1/q} \ \left(\varepsilon + \frac{C_n}{(\log_2 n)^{1/p} \ f(\log_2 n)} \right) \\ &\leq \frac{1}{2} c_{p,q,f}(E) \ \|T\|_1 \ n^{-1/q} \ \left(\varepsilon + \frac{C_n}{(\log_2 n)^{1/p} \ f(\log_2 n)} \right) \ . \end{aligned}$$

To estimate $e_r(S)$ we first observe that

$$\begin{aligned} \|S\| &\leq \|T - A\| + \|T - B\| \\ &\leq 2 \ \varepsilon \ \|T\|_1 + 2 \ (\varepsilon + \delta) \ \|T\|_1 \\ &\leq 4 \ \|T\|_1 \ \left(\varepsilon + \frac{C_n}{(\log_2 M)^{1/p} \ f(\log_2 M)} \right) \\ &\leq 2^{2+1/p+\sigma} \ \|T\|_1 \ \left(\varepsilon + \frac{C_n}{(\log_2 n)^{1/p} \ f(\log_2 n)} \right) \end{aligned}$$

by Lemma 2.1. Since im $S \overset{1}{\hookrightarrow} \ell_\infty^m \overset{1}{\hookrightarrow} \ell_\infty^n$, we finally obtain

$$\begin{aligned} e_r(S) &\leq 2 \ c_q(E) \left(\frac{\log_2 \left(\frac{n}{r} + 1 \right)}{r} \right)^{1/q} \ \|S\| \\ &\leq 2^{3+1/p+\sigma} \ c_q(E) \left(\frac{\log_2 4}{\frac{n}{3}} \right)^{1/q} \ \|T\|_1 \ \left(\varepsilon + \frac{C_n}{(\log_2 n)^{1/p} \ f(\log_2 n)} \right) \\ &\leq \frac{1}{2} c_{p,q,f}(E) \ \|T\|_1 \ n^{-1/q} \ \left(\varepsilon + \frac{C_n}{(\log_2 n)^{1/p} \ f(\log_2 n)} \right) \end{aligned}$$

since E is of entropy cotype q. ◀

Proof of Theorem 3.2: As in the proof of Theorem 3.1 it suffices to consider the case $\varepsilon_1(K) \geq 1$. For fixed p, q and f we let $\gamma := \frac{p}{q(1-\sigma p)}$ and choose an integer m with $m > 2^{\frac{1}{1/p-\sigma}}$. Additionally, for a fixed integer n with $n \geq a := \max\{2^m, m^{3/\gamma}\}$ we define

$$\begin{aligned} C &:= \sup_{k \leq n^\gamma \log_2(n+1)} k^{1/p} \ f(k) \ e_k(K) \\ r &:= \lfloor \gamma \log_m n \rfloor - 1 \\ \alpha_i &:= n^{m^i} && \text{for } i = 0, \ldots, r \\ \beta_i &:= 2^\sigma C \ (\log_2 \alpha_i)^{-1/p} (f(\log_2 \alpha_i))^{-1} && \text{for } i = -1, \ldots, r \ . \end{aligned}$$

An easy computation shows $\lfloor \log_2 \alpha_i \rfloor + 1 \le n^\gamma \log_2(n+1)$ and hence we get

$$\begin{aligned}
\varepsilon_{\alpha_i}(K) &\le C\ (\lfloor \log_2 \alpha_i \rfloor + 1)^{-1/p}\ (f(\lfloor \log_2 \alpha_i \rfloor + 1))^{-1} \\
&< C\ (\log_2 \alpha_i)^{-1/p}\ \left(\frac{\lfloor \log_2 \alpha_i \rfloor + 1}{\log_2 \alpha_i} \right)^\sigma\ (f(\log_2 \alpha_i))^{-1} \\
&\le 2^\sigma C\ (\log_2 \alpha_i)^{-1/p} (f(\log_2 \alpha_i))^{-1}\ =\ \beta_i
\end{aligned}$$

for $0 \le i \le r$. Furthermore, by the definition of m we obtain

$$\frac{2\ \beta_i}{\beta_{i-1}}\ =\ 2\ m^{-1/p}\ \frac{f(m^{i-1}\ \log_2 n)}{f(m^i\ \log_2 n)}\ \le\ 2\ m^{-(1/p-\sigma)}\ \le\ 1\ .$$

Therefore, by Lemma 2.4 we can decompose T in

$$T\ =\ T_o\ +\ \sum_{i=1}^r T_i\ +\ S$$

and receive:

$$e_{2n}(T)\ \le\ e_n(T_o)\ +\ e_n(\sum_{i=1}^r T_i)\ +\ e_1(S)\ . \tag{3.3}$$

First we estimate the term $e_n(T_o)$. By Lemma 2.4 the operator T_o is constructed by a partition of unity $(\psi_i) \subset C(K)$ of at most $\alpha_o = n$ functions with

$$\begin{aligned}
\varepsilon_1(\text{supp}\ \psi_i) &\le \beta_{-1}\ =\ 2^\sigma\ C\ m^{1/p}\ (\log_2 n)^{-1/p}\ (f(m^{-1}\ \log_2 n))^{-1} \\
&\le c_1\ C\ (\log_2 n)^{-1/p}\ (f(\log_2 n))^{-1}\ ,
\end{aligned}$$

where $c_1 := 2^\sigma\ m^{1/p+\sigma}$. Hence by Lemma 3.3 there exists a constant c_2 such that

$$\begin{aligned}
e_n(T_o) &\le c_2\ \|T\|_1\ C\ n^{-1/q}\ (\log_2 n)^{-1/p}\ (f(\log_2 n))^{-1} \\
&\le c_2\ \|T\|_1\ C\ n^{-1/q}\ (\log_2 n)^{-1/p+1/q}\ (f(\log_2 n))^{-1}\ .
\end{aligned}$$

Now we discuss $e_1(S)$. By Lemma 2.4 we know that

$$\begin{aligned}
e_1(S) &\le 2\ \beta_{r-1}\ \|T\|_1 \\
&= 2^{1+\sigma}\ C\ \|T\|_1\ m^{-1/p(r-1)}\ (\log_2 n)^{-1/p}\ (f(m^{r-1}\ \log_2 n))^{-1} \\
&\le 2^{1+\sigma}\ C\ \|T\|_1\ m^{-1/p(r-1)}\ (\log_2 n)^{-1/p}\ m^{\sigma(r-1)}\ (f(\log_2 n))^{-1} \\
&\le c_3\ C\ \|T\|_1\ n^{-1/q}\ (\log_2 n)^{-1/p+1/q}\ (f(\log_2 n))^{-1}\ ,
\end{aligned}$$

where $c_3 := 2^{1+\sigma}\ m^{3(1/p-\sigma)}$. Finally we estimate $e_n(\sum_{i=1}^r T_i)$. For $s := \frac{q}{1+q}$ and suitable $c_4, c_5 \ge 1$ we obtain

$$
\begin{aligned}
& n^{1/q}\ e_n(\sum_{i=1}^{r} T_i) \\
\leq\ & c_4 \left(\sum_{i=1}^{r} (\lambda_{q,\infty}^{(e)}(T_i))^s \right)^{1/s} \\
\leq\ & c_5\ \|T\|_1 \left(\sum_{i=1}^{r} \left((\log_2 \alpha_i)^{1/q}\ \beta_{i-2} \right)^s \right)^{1/s} \\
=\ & 2^{\sigma}\ m^{2/p}\ c_5\ \|T\|_1\ C \left(\sum_{i=1}^{r} \left(m^{-i(1/p-1/q)}\ (\log_2 n)^{1/q-1/p} (f(m^{i-2} \log_2 n))^{-1} \right)^s \right)^{1/s} .
\end{aligned}
$$

The last sum can be estimated by

$$
\begin{aligned}
& \left(\sum_{i=1}^{r} \left(m^{-i(1/p-1/q)}\ (\log_2 n)^{1/q-1/p} (f(m^{i-2} \log_2 n))^{-1} \right)^s \right)^{1/s} \\
\leq\ & m^{\sigma}\ (\log_2 n)^{1/q-1/p} \left(\sum_{i=1}^{r} m^{-i(1/p-1/q)s}\ (f(m^{i-1} \log_2 n))^{-s} \right)^{1/s} \\
\leq\ & (\log_2 n)^{1/q-1/p} \left(\sum_{i=1}^{r} (m^{-i(1/p-1/q-\sigma)s}\ (f(\log_2 n))^{-s} \right)^{1/s} \\
\leq\ & c_6\ (\log_2 n)^{1/q-1/p}\ (f(\log_2 n))^{-1}\ ,
\end{aligned}
$$

where $c_6 := \left(\sum_{i=1}^{\infty} m^{-i(1/p-1/q-\sigma)s} \right)^{1/s}$. Hence we have proven

$$
e_n(\sum_{i=1}^{r} T_i)\ \leq\ c_7\ \|T\|_1\ C\ n^{-1/q}\ (\log_2 n)^{1/q-1/p}\ (f(\log_2 n))^{-1}
$$

for some constant $c_7 > 0$. Therefore we finally receive

$$
n^{1/q} (\log_2 n)^{1/p-1/q} f(\log_2 n)\ e_{2n}(T)\ \leq\ c_8\ \|T\|_1 \sup_{k \leq n^{\gamma} \log_2(n+1)} k^{1/p}\ f(k)\ e_k(K)
$$

for all $n \geq a$ and suitable $c_8 \geq 1$. For $n \leq a$ the assertion is trivial. ◀

3.3 The case of polynomial decay

In this section we will prove an inequality which corresponds with the remaining case of essentially polynomial decay of $(\varepsilon_n(K))$. It complements the result of Carl, Heinrich and Kühn described in the introduction.

Theorem 3.4 *Let E be a Banach space of weak entropy cotype q, $2 \leq q < \infty$. Then for all $p > 0$ and $\gamma \geq 0$ there exists a constant $c \geq 1$, such that for all compact metric spaces (K, d) and all 1-Hölder-continuous operators $T : E \to C(K)$ we have*

$$\sup_{k \leq n} k^{\frac{1}{p}+\frac{1}{q}} \ (\log_2(k+1))^{\gamma} \ e_k(T) \ \leq \ c \ c_K \ \|T\|_1 \sup_{k \leq n^{1+\frac{p}{q}}} k^{\frac{1}{p}} \ (\log_2(k+1))^{\gamma} \ \varepsilon_k(K) \ .$$

We will see in Corollary 3.6 how this theorem can be applied in order to obtain good estimates of $(e_n(T))$ whenever the sequence $(\varepsilon_n(K))$ decreases essentially polynomially. In Proposition 5.10 we show that these estimates are asymptotically optimal for all Banach spaces which are not of better weak entropy cotype than q. Moreover, we will prove in Proposition 5.5 that the above inequality *characterizes* Banach spaces of weak entropy cotype q.
Before we prove Theorem 3.4, we remark that

$$\sum_{i=1}^{n} i^{-b} \ e^{a \cdot i} \ \leq \ \frac{e^a}{a-b} \ (n+1)^{-b} \ e^{a \cdot n} \tag{3.4}$$

holds for all $0 < b < a$ and $n \in \mathbb{N}$.

Proof of Theorem 3.4: Again it suffices to consider the case $\varepsilon_1(K) \geq 1$. For fixed $p > 0$ and $\gamma \geq 0$ we choose an integer a with $a \geq 2 + \max\{16^q, 4^p\}$. Then for a fixed integer $n \geq a$ we define:

$$\begin{aligned}
C &:= \sup_{j \leq n^{1+\frac{p}{q}}} j^{1/p} \ (\log_2(j+1))^{\gamma} \ \varepsilon_j(K) \\
r &:= \lfloor (1/p + 1/q) \log_2(n-1) \rfloor - 3 \\
L &:= \left\lfloor \frac{1}{p} \log_2(n-1) \right\rfloor \\
\alpha_i &:= \lfloor 2^{(i+3)p} + 1 \rfloor \qquad\qquad \text{for } i = 0, \ldots, r \\
\beta_i &:= \max\{1, 1/p^{\gamma}\} \ C \ 2^{-(i+3)} \ (i+3)^{-\gamma} \qquad \text{for } i = -2, \ldots, r \ .
\end{aligned}$$

Clearly, $a \leq n$ implies $1 \leq L \leq r$. Furthermore, since $\alpha_i \leq n^{1+p/q}$, we obtain $\varepsilon_{\alpha_i}(K) < \beta_i$ for $0 \leq i \leq r$. Hence we can decompose T by Lemma 2.4 into

$$T \ = \ \sum_{i=0}^{L-1} T_i \ + \ \sum_{i=L}^{r} T_i \ + \ S$$

and receive

$$e_n(T) \ \leq \ e_{\lfloor \frac{n}{2} \rfloor}(\sum_{i=0}^{L-1} T_i) \ + \ e_{\lfloor \frac{n}{2} \rfloor}(\sum_{i=L}^{r} T_i) \ + \ \|S\| \ . \tag{3.5}$$

Before estimating the terms of the right hand side of inequality (3.5), we observe that $4\beta_{-2} = 4 \max\{1, p^{-\gamma}\}\ C\ 2^{-1} \geq 2$ and thus $\|T_o\| \leq \|T\|_1 \leq 4\ \beta_{-2}\ \|T\|_1$. Hence for $\sigma < q$ and $0 \leq i \leq r$ we obtain

$$\begin{aligned} \|T_i\|\ \alpha_i^{1/\sigma - 1/q} &\leq 4\ \beta_{i-2}\ \|T\|_1\ \lfloor 2^{(i+3)p} + 1 \rfloor^{1/\sigma - 1/q} \\ &\leq c_1\ \|T\|_1\ C\ (i+1)^{-\gamma}\ 2^{(i+1)(p(\frac{1}{\sigma} - \frac{1}{q}) - 1)}\ , \end{aligned} \tag{3.6}$$

where $c_1 := 2^{2+(2p+1)(\frac{1}{\sigma} - \frac{1}{q})} \max\{1, p^{-\gamma}\}$.

To estimate the first term of inequality (3.5), we choose σ such that $\frac{1}{\sigma} > (\frac{\gamma}{\ln 2} + 1)\frac{1}{p} + \frac{1}{q}$ and let $s := \frac{\sigma}{1+\sigma}$. Then by our assumption on E and inequality (3.6) we obtain

$$\begin{aligned} \left\lfloor \frac{n}{2} \right\rfloor^{1/\sigma} e_{\lfloor \frac{n}{2} \rfloor}(\sum_{i=0}^{L-1} T_i) &\leq c_2 \left(\sum_{i=0}^{L-1} \left(\lambda_{\sigma,\infty}^{(e)}(T_i) \right)^s \right)^{1/s} \\ &\leq c_2\ c_3 \left(\sum_{i=0}^{L-1} \left(\|T_i\|\ \alpha_i^{1/\sigma - 1/q} \right)^s \right)^{1/s} \\ &\leq c_4\ C\ \|T\|_1 \left(\sum_{i=1}^{L} i^{-\gamma s}\ 2^{is(p(\frac{1}{\sigma} - \frac{1}{q}) - 1)} \right)^{1/s} \end{aligned} \tag{3.7}$$

for suitable $c_2, c_3 \geq 1$ and $c_4 := c_1\ c_2\ c_3$. By inequality (3.4) there exists a constant $c_5 > 0$ such that we may conclude:

$$\begin{aligned} e_{\lfloor \frac{n}{2} \rfloor}(\sum_{i=0}^{L-1} T_i) &\leq 3^{1/\sigma}\ c_4\ C\ \|T\|_1\ n^{-1/\sigma} \left(\sum_{i=1}^{L} i^{-\gamma s}\ 2^{is(p(\frac{1}{\sigma} - \frac{1}{q}) - 1)} \right)^{1/s} \\ &\leq 3^{1/\sigma}\ c_4\ c_5\ C\ \|T\|_1\ n^{-1/\sigma}\ (L+1)^{-\gamma}\ 2^{L\ (p(\frac{1}{\sigma} - \frac{1}{q}) - 1)} \\ &\leq c_6\ C\ \|T\|_1\ n^{-1/\sigma} \left(\frac{1}{p} \log_2(n-1) \right)^{-\gamma} 2^{\frac{1}{p}\log_2(n-1)\ (p(\frac{1}{\sigma} - \frac{1}{q}) - 1)} \\ &\leq (4p)^{\gamma}\ c_6\ C\ \|T\|_1\ (\log_2(n+1))^{-\gamma}\ n^{-1/p - 1/q}\ , \end{aligned}$$

where $c_6 := 3^{1/\sigma}\ c_4\ c_5$.

Next we estimate the second term of inequality (3.5). For some fixed $\sigma < p$ with $\frac{1}{\sigma} < \frac{1}{q} + \frac{1}{p}$ and $s := \frac{\sigma}{1+\sigma}$ we get analogously to (3.7):

$$\left\lfloor \frac{n}{2} \right\rfloor^{1/\sigma} e_{\lfloor \frac{n}{2} \rfloor}(\sum_{i=L}^{r} T_i) \leq c_7\ C\ \|T\|_1 \left(\sum_{i=L}^{r} (i+1)^{-\gamma s}\ 2^{(i+1)\ s\ (p(\frac{1}{\sigma} - \frac{1}{q}) - 1)} \right)^{1/s}\ ,$$

where $c_7 > 0$ is a suitable constant. Thus we obtain

$$\begin{aligned} e_{\lfloor \frac{n}{2} \rfloor}(\sum_{i=L}^{r} T_i) &\leq 3^{1/\sigma}\ c_7\ C\ \|T\|_1\ n^{-1/\sigma} \left(\sum_{i=L}^{r} (i+1)^{-\gamma s}\ 2^{(i+1)\ s\ (p(\frac{1}{\sigma} - \frac{1}{q}) - 1)} \right)^{1/s} \\ &\leq c_8\ C\ \|T\|_1\ n^{-1/\sigma}\ (L+1)^{-\gamma}\ 2^{L\ (p(\frac{1}{\sigma} - \frac{1}{q}) - 1)} \\ &\leq 2^{1+1/p+2\gamma}\ p^{\gamma}\ c_8\ C\ \|T\|_1\ (\log_2(n+1))^{-\gamma}\ n^{-1/p-1/q} \end{aligned}$$

for some constant $c_8 \geq 1$. Finally we consider the last term. By Lemma 2.4 we receive

$$\begin{aligned} \|S\| &\leq 2\ \beta_{r-1}\ \|T\|_1 \\ &\leq 2^{3+\gamma}\ \max\{1, p^{-\gamma}\}\ C\ \|T\|_1\ (r+4)^{-\gamma}\ 2^{-(r+4)} \\ &\leq c_9\ C\ \|T\|_1\ (\log_2(n+1))^{-\gamma}\ n^{-1/p-1/q}\ , \end{aligned}$$

where $c_9 := 2^{3+1/p+1/q+3\gamma} \cdot (1/p+1/q)^{-\gamma} \cdot \max\{1, p^{-\gamma}\}$. Combining the estimates we easily get the assertion. ◀

3.4 Some consequences

We now want to illustrate how the proven inequalities can be used to obtain estimates of $(e_n(T))$, if one has estimates for $(\varepsilon_n(K))$. We begin with the case of slowly decreasing sequences $(\varepsilon_n(K))$:

Corollary 3.5 *Let E be a Banach space of entropy cotype q, $2 \leq q < \infty$ and f be a σ-controlled function. Then for all $0 < p \leq \infty$ with $p \neq q$, all compact metric spaces (K, d) with*

$$e_n(K) \ \preceq\ n^{-1/p}\ f(\log_2(n+1))$$

and all 1-Hölder-continuous operators $T : E \to C(K)$ we have

$$e_n(T) \ \preceq\ n^{-1/p}\ f(\log_2(n+1))$$

in the case of $q < p \leq \infty$ and

$$e_n(T) \ \preceq\ n^{-1/q}\ (\log_2(n+1))^{1/q-1/p}\ f(\log_2(\log_2(n+1)+1))$$

in the case of $0 < p < q$.
In particular, these estimates hold if $f(t) = t^{\pm\sigma}$.

We will see in Proposition 5.8 that the first estimate is asymptotically optimal for some 1-Hölder-continuous operator $T : E \to C(K)$ if $q < p \leq \infty$. If E is not of any entropy cotype $q - \varepsilon$, then it is also asymptotically optimal for some 1-Hölder-continuous operator $T : E \to C(K)$ in the case of $0 < p < q$. This will be shown in Proposition 5.9.

Proof: Let $c_1 > 1$ and $g(t) := f(\log_2(c_1 t + 1))$. Then one easily checks that g is $\frac{\sigma}{\ln c_1}$-controlled. Moreover we obtain

$$c_1^{-\sigma}\ f(\log_2(t+1)) \ \leq\ g(t) \ \leq\ c_1^{\sigma}\ f(\log_2(t+1))$$

for all $t \geq 3$. Hence if we choose $c_1 > 0$ large enough, we can apply Theorem 3.2 to prove the case $0 < p < q$.
To show the case $q < p \leq \infty$ we choose $c_1 > 1$ such that for $\tau := \frac{\sigma}{\ln c_1}$ we have $\frac{1}{p} + \tau < \frac{1}{q}$. Moreover, we take $\tilde{p} \in (q, \infty)$ such that $\frac{1}{\tilde{p}} > \frac{1}{p} + \tau$. Now by Theorem 3.1 we obtain

$$\begin{aligned} n^{\frac{1}{\tilde{p}}}\ e_n(T) &\leq c_2\ c_K\ \|T\|_1\ \sup_{k \leq n} k^{\frac{1}{\tilde{p}}}\ e_k(K) \\ &\leq \rho\ c_1^{\sigma}\ c_2\ c_K\ \|T\|_1\ \sup_{k \leq n} k^{\frac{1}{\tilde{p}} - \frac{1}{p}}\ g(k) \\ &\leq \rho\ c_1^{\sigma}\ c_2\ c_K\ \|T\|_1\ \sup_{k \leq n} k^{\frac{1}{\tilde{p}} - \frac{1}{p}}\ \left(\frac{n}{k}\right)^{\tau}\ g(n) \\ &\leq \rho\ c_1^{2\sigma}\ c_2\ c_K\ \|T\|_1\ n^{\frac{1}{\tilde{p}} - \frac{1}{p}}\ f(\log_2(n+1))\ . \blacktriangleleft \end{aligned}$$

If one has a compact metric space whose entropy numbers essentially decrease like $n^{-1/p}$, the following corollary can be applied. It complements results of [9] and [24].

Corollary 3.6 *Let E be a Banach space of weak entropy cotype q, $2 \leq q < \infty$ and f be a σ-controlled function. Then for all $0 < p < \infty$ there is a constant $c \geq 1$ such that for all compact metric spaces (K, d) with*

$$\varepsilon_n(K) \leq \rho\ n^{-1/p}\ f(\log_2(n+1))\ , \qquad n \in \mathbb{N}$$

and all 1-Hölder-continuous operators $T : E \to C(K)$ we have

$$e_n(T) \leq c\ \rho\ c_K\ \|T\|_1\ n^{-1/p-1/q}\ f(\log_2(n+1))\ , \qquad n \in \mathbb{N}.$$

In particular, this holds if $f(t) = t^{\pm\sigma}$.

We will see in Proposition 5.10 that the above estimate is asymptotically optimal for some 1-Hölder-continuous $T : E \to C(K)$ if E is not of any weak entropy cotype $q - \varepsilon$.

Proof: With the help of Theorem 3.4 we obtain

$$\begin{aligned} & n^{\frac{1}{p}+\frac{1}{q}}\ (\log_2(n+1))^{\sigma}\ e_n(T) \\ \leq\ & c_1 \cdot c_K\ \|T\|_1\ \sup_{k \leq n^{1+\frac{p}{q}}} k^{\frac{1}{p}}\ (\log_2(k+1))^{\sigma} \varepsilon_k(K) \\ \leq\ & \rho\ c_1 \cdot c_K\ \|T\|_1\ \sup_{k \leq n^{1+\frac{p}{q}}} (\log_2(k+1))^{\sigma} f(\log_2(k+1)) \\ \leq\ & \rho\ c_2 \cdot c_K\ \|T\|_1\ \sup_{k \leq n^{1+\frac{p}{q}}} (\log_2(k+1))^{\sigma} \left(\frac{\log_2((n+1)^{1+\frac{p}{q}})}{\log_2(k+1)}\right)^{\sigma} f(\log_2((n+1)^{1+\frac{p}{q}})) \\ \leq\ & \rho\ c_3 \cdot c_K\ \|T\|_1\ (\log_2(n+1))^{\sigma}\ f(\log_2(n+1))\ . \blacktriangleleft \end{aligned}$$

Examples of compact subsets $K \subset \mathbb{R}^n$ with

$$\varepsilon_n(K) \ \sim \ n^{-1/p} \, (\log_2(n+1))^{\gamma} \cdot \ldots$$

can be found in [13, Sect. 4]. Their construction is based on the idea used for the Cantor set.

Remark 3.7 Corollary 3.6 states in particular that, given a compact metric space (K,d) with $(\varepsilon_n(K)) \in \ell_{p,\infty}$ and a Banach spaces E of entropy cotype q, we have $(e_n(T)) \in \ell_{s,\infty}$ for every 1-Hölder-continuous operator $T : E \to C(K)$ and $1/s := 1/p + 1/q$. But what happens if we have $(\varepsilon_n(K)) \in \ell_{p,r}$? A partial answer can be given with Theorem 3.4:

Let E be a Banach space of weak entropy cotype q and let $0 < p, r < \infty$. Then there is a constant $c_{p,r} > 0$ such that for every compact metric space (K,d) with

$$2^{1/p} \, e_{n+1}(K) \ \le \ e_n(K) \ , \qquad\qquad n \ge 1 \tag{3.8}$$

and all 1-Hölder-continuous operators $T : E \to C(K)$ we have

$$\sum_{k=1}^{n} \bigl(k^{1/p+1/q-1/r} \ e_k(T)\bigr)^r \ \le \ c_{p,r} \ c_K \ \|T\|_1 \ \sum_{k=1}^{n^{1+p/q}} \bigl(k^{1/p-1/r} \ \varepsilon_k(K)\bigr)^r$$

In particular, if $(\varepsilon_n(K)) \in \ell_{p,r}$ and condition (3.8) holds then $(e_n(T)) \in \ell_{s,r}$ with $\frac{1}{s} = \frac{1}{p} + \frac{1}{q}$.

To see this we let $a_n := n^{1/s} \, e_n(T)$ and $s_n := n^{1/p} \, \varepsilon_n(K)$. Then for $0 < t < \infty$ and suitable constants $c_1, c_2 > 0$ we have

$$\begin{aligned} n^{1/t} \ a_{2^n} \ &= \ n^{1/t} \ 2^{n/s} \ e_{2^n}(T) \\ &\le \ c_1 \sup_{k \le 2^{n(1+p/q)}} (\log_2(k+1))^{1/t} \ k^{1/p} \ \varepsilon_k(K) \\ &\le \ c_2 \sup_{k \le (1+p/q)n} k^{1/t} \ 2^{k/p} \ \varepsilon_{2^k}(K) \\ &= \ c_2 \sup_{k \le (1+p/q)n} k^{1/t} \ s_{2^k} \ . \end{aligned}$$

Now condition (3.8) guarantees that (s_{2^k}) is decreasing. Therefore we obtain

$$a_{2^n} \ \le \ c_2 \left(\frac{\sum_{k=1}^{(1+p/q)n} (s_{2^k})^t}{n} \right)^{1/t} \ . \tag{3.9}$$

An easy modification of Hardy's inequality (cf. [12, Lem. 1.5.3.]) together with the argument used in the proof of [12, Th. 3.1.2.] yields

$$\sum_{k=1}^{n} (a_{2^k})^r \ \le \ c_3 \sum_{k=1}^{n^{1+p/q}} (s_{2^k})^r \ ,$$

but this is equivalent to the assertion.
We conjecture that condition (3.8) is superfluous. One might achieve this by proving (3.9) directly.

3.5 Some remarks and further generalizations

Remark 3.8 As mentioned in Remark 1.30, there is a constant $c > 0$ such that for every operator $T : H \to \ell_\infty^n$ starting in a Hilbert space H we have

$$d_k(T) \;\leq\; c\; \left(\frac{\log_2(\frac{n}{k}+1)}{k}\right)^{1/2} \;\|T\|\;, \qquad\qquad 1 \leq k \leq n\;.$$

In particular such estimates hold for the Tichomirov numbers $t_k(T)$. Considering the proofs of Theorems 3.1, 3.2 and 3.4, we also observe that we only used some of the usual s-number properties of the entropy numbers plus their injectivity. Moreover, for compact operators $T : E \to C(K)$ or $T : E \to \ell_\infty(A)$ the Tichomirov numbers coincide with the Kolmogorov numbers (cf. [12, Th. 5.3.2.] and [12, Th. 2.2.1., Prop. 2.3.3. and (2.6.3.)]). Therefore we can restate Theorems 3.1, 3.2 and 3.4 using Kolmogorov numbers instead of entropy numbers, if we start in a Hilbert space and let $q = 2$.
Moreover, in the situation of Theorem 3.4 we can also employ estimate (1.2) while repeating the proof carefully. We then obtain the following result:

Let E be an $\mathcal{L}_q$-space for some $1 < q < \infty$ and $q^* := \max\{q, q'\}$. Then for all $p > 0$ with $\frac{1}{2} < \frac{1}{q^*} + \frac{1}{p}$ and every $\gamma \geq 0$ there exists a constant $c \geq 1$, such that for all compact metric spaces (K,d) and all 1-Hölder-continuous operators $T : E \to C(K)$ we have

$$\sup_{k\leq n} k^{\frac{1}{p}+\frac{1}{q^*}}\; (\log_2(k+1))^\gamma\; d_k(T) \;\leq\; c\; c_K\; \|T\|_1 \sup_{k \leq n^{1+\frac{p}{q^*}}} k^{\frac{1}{p}}\; (\log_2(k+1))^\gamma\; \varepsilon_k(K)\;.$$

In particular, if (K,d) is a metric space with $\varepsilon_n(K) \preceq n^{-1/p}\; f(\log(n+1))$ for some real number $0 < p \leq 2$ and some σ-controlled function f, then for every 1-Hölder-continuous operator $T : E \to \ell_\infty(K)$ starting in some $\mathcal{L}_q$-space, $1 < q < \infty$, we have

$$d_n(T) \;\preceq\; \|T\|_1\; n^{-\frac{1}{q^*}-\frac{1}{p}}\; f(\log_2(n+1))\;.$$

This can be especially applied for compact subsets $K \subset \mathbb{R}^2$.

It is an open problem to determine the behaviour of $(e_n(T))$ in the case of $(e_n(K)) \in \ell_{q,\infty}$ and $T : E \to C(K)$ being a 1-Hölder-continuous operator that starts in a Banach space of entropy cotype q, i.e. in the limiting case $p = q$ of Corollary 3.5. With the proof of Theorem 3.1 we obtain

$$\sup_{k \le n} k^{1/q} \, (\log(k+1))^{-(1+1/q)} \, e_k(T) \; \le \; c \, c_K \, \|T\|_1 \sup_{k \le n} k^{1/q} \, e_k(K) \, , \tag{3.10}$$

but we conjecture that the extra log-factor can be dropped. Moreover, if we know that $(e_n(K))$ decreases slightly faster than $n^{-1/q}$, say $e_n(K) \preceq n^{-1/q} \, (\log_2(k+1))^{-\gamma}$ for some $\gamma > 1 + 1/q$, we can apply

$$\sup_{k \le n} k^{\frac{1}{q}} (\log_2(\log_2(k+1)+1))^{\gamma-(1+\frac{1}{q})} \, e_k(T) \; \le \; c \, c_K \, \|T\|_1 \sup_{k \le n} k^{\frac{1}{q}} (\log_2(k+1))^{\gamma} \, e_k(K)$$

which can be proven analogously to Theorem 3.2. In this case one can see, using the techniques for Proposition 5.9, that this is asymptotically optimal apart from the factor $(\log_2(\log_2(k+1)+1))^{-(1+\frac{1}{q})}$.

The theorems of this chapter are inspired by the work of Carl, Heinrich and Kühn in [9], where a simpler form of Corollary 3.6 has been proven. They then applied their result to certain integral operators defined by so-called Hölder-continuous kernels (cf. [12, Ch. 5]). Eigenvalue estimates of such operators can be found in [20], [25] and with some weaker condition on the kernel in [13].
Operators $T_k : E \to \ell_\infty(X)$ defined by so-called abstract kernels $k : X \to E'$, where k is bounded (cf. [12, Ch. 5.13]), were considered by Edmunds and Carl in [8]. To apply the results of [9] to such operators they defined a metric d on X such that T_k became 1-Hölder-continuous. The entropy numbers of (X, d) then coincide with those of im k. Therefore one can estimate the entropy numbers of T_k with the help of those of im k. This idea was used in [18] firstly. We pick-up the idea of *making* an operator 1-Hölder-continuous in Lemma 5.1 in order to show that every operator $T : E \to F$ shares its entropy numbers with a suitable 1-Hölder-continuous operator $T : E \to \ell_\infty(K)$.
Finally we wish to remark that Carl, Kyrezi and Pajor applied the result of [9] to estimate the entropy numbers of a precompact, absolutely convex set in terms of the entropy numbers of its extremal points. This will be done with our inequalities in the following chapter.

Chapter 4

Entropy of convex hulls

Given a precompact subset A of a Banach space E, we know that the absolutely convex hull $\mathrm{co}A$ of A is again precompact. If we additionally know something about the behaviour of $(\varepsilon_n(A))$, it is rather natural to ask for the behaviour of $(e_n(\mathrm{co}A))$. If we consider an arbitrary Banach space E, it was proved by Carl, Kyrezi and Pajor in [10, Prop. 4.4] that for every precompact subset $A \subset E$ and all $0 < p < \infty$

$$\sup_{k \le n} \; k^{1/p} \; e_k(\mathrm{co}A) \;\le\; c_p \; c_A \; \sup_{k \le n} \; k^{1/p} \; \varepsilon_k(A) \tag{4.1}$$

hold, where $c_p > 0$ is a constant only depending on p and

$$c_A := \frac{\|A\|}{\varepsilon_1(A)} \; ,$$

where $\|A\| := \sup_{x \in A} \|x\|$. Note that $c_A \ge 1$ in general and $c_A = 1$, if A is symmetric, i.e. $A = -A$.
However, if E is B-convex or even a Hilbert space, better estimates are known if $\varepsilon_k(A) \preceq n^{-1/p}$ or $\varepsilon_k(A) \preceq (\log(n+1))^{-1/p}$ (cf. [16],[1],[7] and [10]). To prove the case of polynomial decay, one can use the result of [9, Th. 1] on 1-Hölder-continuous operators $T : E \to \ell_\infty(K)$ which motivated the work of the last chapter (cf. [7] and [10]). Hence it seems to be a good idea to carry over the results of the last chapter onto this problem. Therefore we consider the operator $T_A : \ell_1(A) \to E$ defined by $T_A(e_t) := t$ on the canonical basis $(e_t)_{t \in A}$ of $\ell_1(A)$. Since $\overline{\mathrm{co}A} = \overline{T_A(B_{\ell_1(A)})}$ we have

$$e_n(T_A) \;=\; e_n(\mathrm{co}A) \; .$$

Moreover, T_A' as an operator mapping E' into $\ell_\infty(A)$ is 1-Hölder-continuous with $\|T_A'\|_1 = \max\{\|A\|, 1\}$. Now the ansatz is simple: We use the inequalities of the last chapter to estimate $e_n(T_A')$ and then we apply the duality Theorem 1.18. These scheme will be worked out in sections 4.1, 4.2 and 4.3.
As pointed out in Remark 3.8, Theorems 3.1, 3.2 and 3.4 also hold for the Kolmogorov

numbers if E is a Hilbert space. Hence we also obtain estimates for the Gelfand widths $c_n(\mathrm{co}A) := c_n(T_A)$. Note that if A is finite and $E_o := \mathrm{im}\ T_A$, we have

$$c_n(\mathrm{co}A) = \inf\{\mathrm{diam}\ (\mathrm{co}A \cap F) \ : \ F \subset E_o \text{ subspace with } \dim F > \dim E_o - n\ \} .$$

Hence $c_n(\mathrm{co}A)$ measures the minimal diameter of m-dimensional, $m \geq \dim E_o - n + 1$, slices of $\mathrm{co}A$ in this case. For this interpretation, we also refer to [28].
Moreover, we have to check that the resulting inequalities of our ansatz produce good, i.e. asymptotically optimal estimates. For this we need a technique to construct subsets for which we can control both $(\varepsilon_n(A))$ and $(e_n(\mathrm{co}A))$. In the sequence spaces ℓ_p this is no problem using the canonical basis of them, but what can be done in general Banach spaces? It turns out that the idea of this construction in ℓ_p is in fact of local nature, i.e. it also can be made in Banach spaces containing ℓ_p^n's uniformly. This is the content of the first theorem:

Theorem 4.1 *Let E be an infinite dimensional Banach space which uniformly contains ℓ_p^n's for some $1 \leq p < \infty$ and let (a_n) be a regular sequence. Then there exists a subset A of E and a constant $c > 0$ such that*

$$\varepsilon_n(A) \ \sim \ a_n$$

and for all $n, m \in \mathbb{N}$ we additionally have

$$e_n(\mathrm{co}A) \ \geq \ c\ a_m\ e_n(id : \ell_1^m \to \ell_p^m) \ .$$

Proof: Let c_1 be a constant such that $a_n \leq c_1\ a_{2n}$ and $a_m \leq c_1\ a_n$ for all $1 \leq n \leq m$. Without loss of generality we may assume that E contains ℓ_p^n's 2-uniformly. Hence for all $n \geq 0$ there is a subspace $E_n \subset E$ and an isomorphism $T_n : E_n \to \ell_p^{2^n}$ with $\|T_n\| = 1$ and $\|T_n^{-1}\| \leq 2$. Denoting by $e_1, \ldots, e_{2^n}$ the canonical basis of $\ell_p^{2^n}$ we define

$$A_n \ := \ \{a_{2^n}\ T_n^{-1} e_j \mid 1 \leq j \leq 2^n\} \ .$$

Moreover we let

$$A \ := \ \bigcup_{n=0}^{\infty} A_n \cup \{0\} \ .$$

We begin with the estimate of $\varepsilon_n(A)$ from above. Let $k := \lfloor \log_2 n \rfloor$, i.e $2^k \leq n < 2^{k+1}$. Hence we have $\varepsilon_n(A) \leq \varepsilon_{2^k}(A)$. To cover $\bigcup_{i=0}^{k-1} A_i$ we need at most $\sum_{i=0}^{k-1} 2^i = 2^k - 1$ balls of arbitrarily small diameter. For the remaining points

$$a_{2^i}\ T_i^{-1} e_j \in \bigcup_{i=k}^{\infty} A_i \qquad\qquad (1 \leq j \leq 2^i)$$

we have

$$\left\|a_{2^i}\ T_i^{-1} e_j\right\| \ \leq \ c_1\ a_{2^k}\ \left\|T_i^{-1} e_j\right\| \ \leq \ 2\ c_1^2\ a_{2^{k+1}} \ \leq \ 2\ c_1^3\ a_n \ .$$

Therefore, we finally obtain $\varepsilon_n(A) \leq \varepsilon_{2^k}(A) \leq 2\ c_1^3\ a_n$.
To estimate $\varepsilon_n(A)$ from below we let $k := \lfloor \log_2 n \rfloor$ again. Then, for $1 \leq j_1, j_2 \leq 2^{k+1}$ we have

$$\left\| a_{2^{k+1}}\ T_{k+1}^{-1} e_{j_1} - a_{2^{k+1}}\ T_{k+1}^{-1} e_{j_2} \right\| \ \geq\ \frac{1}{c_1}\ a_{2^k}\ \|e_{j_1} - e_{j_2}\|_{\ell_p^{2^k}}\ \geq\ \frac{2^{1/p}}{c_1^2}\ a_n\ .$$

Hence A_{k+1} is a $\frac{2^{1/p-1}}{c_1^2}\ a_n-$distant subset of $A_{k+1} \subset E$ consisting of $2^{k+1} > n$ points. Therefore we finally obtain

$$\varepsilon_n(A)\ \geq\ \varepsilon_n(A_{k+1})\ \geq\ \frac{1}{2\ c_1^2}\ a_n\ ,$$

where the inequality of the right hand side can be seen with the help of the inner entropy numbers.
To prove the last inequality we let $k := \lfloor \log_2 m \rfloor$ and consider the operator $D : \ell_1^{2^{k+1}} \to \ell_p^{2^{k+1}}$ defined by $e_i \mapsto a_{2^{k+1}}\ e_i$. We have $\mathrm{co}A_{k+1} = T_{k+1}^{-1}\ D(B_{\ell_1^{2^{k+1}}})$ and since $D = T_{k+1}\ T_{k+1}^{-1}\ D$ we obtain

$$\begin{aligned} e_n(\mathrm{co}A) &\geq e_n(\mathrm{co}A_{k+1}) \\ &\geq e_n(D) \\ &= a_{2^{k+1}}\ e_n(id : \ell_1^{2^{k+1}} \to \ell_p^{2^{k+1}}) \\ &\geq \frac{1}{2\ c_1}\ a_{2^k}\ e_n(id : \ell_1^m \to \ell_p^m) \\ &\geq \frac{1}{2\ c_1^2}\ a_m\ e_n(id : \ell_1^m \to \ell_p^m)\ . \blacktriangleleft \end{aligned}$$

4.1 The case of slow logarithmic decay

In this section we use Theorem 3.1 to estimate $(e_n(\mathrm{co}A))$ in terms of $(\varepsilon_n(A))$. It turns out that we obtain stronger consequences than one might expect considering Corollary 3.5. Moreover, we start investigating the local structure of E in terms of (entropy) type under the assumption of known entropy estimates. In particular we are able to characterize B-convexity by the behaviour of entropy numbers of convex hulls.

Theorem 4.2 *Let E be a Banach space of entropy type $q \in (1, 2]$. Then for all $p \in (q', \infty)$ there exists a constant $c \geq 1$, such that for all precompact $A \subset E$ we have*

$$\sup_{k \leq n}\ k^{1/p}\ e_k(\mathrm{co}A)\ \leq\ c\ c_A\ \sup_{k \leq n}\ k^{1/p}\ e_k(A)\ .$$

If E is a Hilbert space and $q = 2$, this is also true for the Gelfand widths $c_k(\mathrm{co}A)$.

Proof: We consider the metric $d(x,y) := \frac{1}{\varepsilon_1(A)} \|x-y\|$ on A. One easily checks that $e_n((A,d)) = \frac{1}{\varepsilon_1(A)} e_n(A)$ and $|T'_A : E' \to \ell_\infty((A,d))|_1 = \varepsilon_1(A)$. Hence we get $\varepsilon_1((A,d)) = 1$ and $\|T'_A : E' \to \ell_\infty((A,d))\|_1 = \|A\|$. Moreover, E' is of entropy cotype q' and therefore by Theorem 1.18 and Theorem 3.1 we obtain

$$\begin{aligned}
\sup_{k\leq n} k^{1/p}\, e_k(\mathrm{co}A) &= \sup_{k\leq n} k^{1/p}\, e_k(T_A : \ell_1(A) \to E) \\
&\leq c_1 \sup_{k\leq n} k^{1/p}\, e_k(T'_A : E' \to \ell_\infty(A)) \\
&= c_1 \sup_{k\leq n} k^{1/p}\, e_k(T'_A : E' \to \ell_\infty((A,d))) \\
&\leq c_2\, c_A \sup_{k\leq n} k^{1/p}\, e_k(A)
\end{aligned}$$

for suitable constants $c_1, c_2 \geq 1$. If E is a Hilbert space, the assertion for the Gelfand widths follows from $c_n(T_A) = d_n(T'_A)$ and Remark 3.8. ◀

Now we could prove a dual version of Corollary 3.5, but something more can be said. Therefore we observe that in the situation of Theorem 4.2 we have

$$\sup_{k\leq n} k^{1/p} e_k(A) \;\leq\; \sup_{k\leq n} k^{1/p} e_k(\mathrm{co}A) \;\preceq\; \sup_{k\leq n} k^{1/p} e_k(A)$$

and that in the Hilbert space case we have

$$\sup_{k\leq n} k^{1/p} e_k(A) \;\leq\; \sup_{k\leq n} k^{1/p} e_k(\mathrm{co}A) \;\preceq\; \sup_{k\leq n} k^{1/p} c_k(\mathrm{co}A) \;\preceq\; \sup_{k\leq n} k^{1/p} e_k(A)$$

by Theorem 1.4. Hence, with the techniques used in the proof of Corollary 1.19 we obtain:

Corollary 4.3 *Let E be a Banach space of entropy type $q \in (1,2]$ and $p \in (q', \infty)$. If (a_n) is a regular sequence with $a_n \leq 2^{1/p} a_{2n}$, then for every precompact subset $A \subset E$ we have:*

$$e_n(A) \preceq a_n \qquad \textit{if and only if} \qquad e_n(\mathrm{co}A) \preceq a_n$$

and

$$e_n(A) \sim a_n \qquad \textit{if and only if} \qquad e_n(\mathrm{co}A) \sim a_n \;.$$

If E is a Hilbert space, these statements are also equivalent to $c_n(\mathrm{co}A) \preceq a_n$, respectively $c_n(\mathrm{co}A) \sim a_n$.

The above corollary states that in $B-$convex Banach spaces E the subsets A and $\mathrm{co}A$ paradoxically have the same entropy behaviour whenever $(e_n(A))$ or $(e_n(\mathrm{co}A))$ decreases ‘slowly’ in the above sense. This means that both sets have the same degree of compactness in this case! This property is surprising and hard to understand since A can be very small in comparison with $\mathrm{co}A$ as the following example illustrates:

Example 4.4 Let $A := \{(\log_2(\log_2(n+1)+1)^{-1}\ e_n \mid n \in \mathbb{N}\}$ be a subset of ℓ_p, $1 \le p \le 2$. Then one easily checks that

$$e_n(A) \ \sim\ (\log_2(n+1))^{-1}\ .$$

Hence for $1 < p \le 2$ we obtain by Corollary 4.3:

$$e_n(\mathrm{co}A) \ \sim\ (\log_2(n+1))^{-1}\ .$$

In contrast to this it was shown in [10] that in ℓ_1 we have

$$e_n(\mathrm{co}A) \ \sim\ (\log_2(\log_2(n+1)+1)^{-1}\ .$$

Note that ℓ_p is B-convex for $1 < p \le 2$ while ℓ_1 is not and that this B-convexity causes the difference between the estimates for ℓ_p and ℓ_1 in the above example. However, by Theorem 4.1 the construction of the set A can also be made for Banach spaces containing ℓ_1^n's uniformly. Therefore we obtain the following characterization:

Theorem 4.5 *The following statements on a Banach space E are equivalent:*

i) E is not B-convex

ii) For all regular sequences (a_n) there is a subset A of E such that

$$\varepsilon_n(A) \sim a_n \sim e_n(\mathrm{co}A)\ .$$

Proof: $i) \to ii)$: By Pisier's characterization of B-convexity E contains ℓ_1^n's uniformly. Hence by Theorem 4.1 we get a subset A of E such that $\varepsilon_n(A) \sim a_n$ and

$$e_n(\mathrm{co}A) \ \succeq\ a_n\ e_n(id : \ell_1^n \to \ell_1^n) \ \succeq\ a_n\ .$$

The estimate $e_n(\mathrm{co}A) \preceq a_n$ follows by [10, Prop. 4.5] or inequality (4.1).
$ii) \to i)$: Suppose that E is B-convex. We take $a_n := (\log_2(\log_2(n+1)+1))^{-1}$ and apply Corollary 4.3 to see that $ii)$ does not hold. ◀

In particular Theorem 4.5 states that the estimates for $e_n(\mathrm{co}A)$ resulting from inequality (4.1) cannot be improved in non B-convex Banach spaces. However, to characterize B-convexity it suffices to consider the sequence $a_n := (\log_2(\log_2(n+1)+1))^{-1}$ in condition $ii)$. Several other types of regular sequences are also possible, e.g. $a_n = n^{-1/p}$ or $a_n = (\log_2(n+1))^{-1/p}$, as one can check with the results of the following sections. Therefore, one might guess that condition $ii)$ can be replaced by

ii') There is a precompact subset A of E such that $(\varepsilon_n(A))$ is regular and

$$\varepsilon_n(A) \sim e_n(\mathrm{co}A)\ .$$

But condition $ii')$ does not characterize B-convexity as the following example shows:

Example 4.6 *Let E be an arbitrary Banach space and (a_n) be a regular sequence with $a_n \sim a_{2^n}$ (cf. Example 1.3). Then there is a precompact subset A of E such that*

$$\varepsilon_n(A) \;\sim\; a_n \;\sim\; e_n(\mathrm{co}A) \; .$$

To see this we remember that E contains ℓ_2^n's uniformly by Dvoretzky's Theorem. Therefore by Theorem 4.1 we find a subset A of E with $\varepsilon_n(A) \sim a_n$ and

$$e_n(\mathrm{co}A) \;\geq\; c\; a_{2^n}\; e_n(id : \ell_1^{2^n} \to \ell_2^{2^n}) \;\geq\; c\; a_{2^n}\; \left(\frac{\log_2(\frac{2^n}{n}+1)}{n}\right)^{1/2} \;\succeq\; a_n \; .$$

To estimate $e_n(\mathrm{co}A)$ from above we use [10, Prop. 4.5] or inequality (4.1) to obtain $e_n(\mathrm{co}A) \preceq a_n$.

Again we consider the phenomenon that in B-convex spaces the subsets A and $\mathrm{co}A$ have the same entropy behaviour whenever one of them decreases 'slowly'. One might ask whether such a phenomenon also exists in non B-convex spaces. Of course Theorem 4.5 states that this cannot be true, e.g. for decay of iterated logarithmic type, i.e. of type

$$(\log(\ldots\log(n+1)+\ldots)+1)^{-1/p} \; .$$

However the following proposition shows that such a phenomenon indeed exists provided one considers 'very slowly' decreasing sequences:

Proposition 4.7 *Let E be an arbitrary Banach space and (a_n) be a regular sequence with $a_n \sim a_{2^n}$. Then for every precompact subset $A \subset E$ we have:*

$$e_n(A) \preceq a_n \qquad \textit{if and only if} \qquad e_n(\mathrm{co}A) \preceq a_n$$

and

$$e_n(A) \sim a_n \qquad \textit{if and only if} \qquad e_n(\mathrm{co}A) \sim a_n \; .$$

Proof: The first equivalence follows from [10, Prop. 4.5] or inequality (4.1) and the trivial fact $e_n(A) \leq e_n(\mathrm{co}A)$. For the second one we first assume $e_n(A) \sim a_n$. Then we already know $e_n(\mathrm{co}A) \preceq a_n$ and trivially we also have $e_n(\mathrm{co}A) \geq e_n(A) \sim a_n$. To prove the converse implication we observe that $e_n(\mathrm{co}A) \sim a_n$ implies:

$$\sup_{k\leq n} k^{1/p}\varepsilon_k(A) \;\preceq\; \sup_{k\leq n} k^{1/p} e_k(\mathrm{co}A) \;\preceq\; \sup_{k\leq n} k^{1/p}\varepsilon_k(A) \; .$$

Then the trick used for Corollary 1.19 yields the assertion. ◀

Suppose now that we know for a given Banach space E that $e_n(A) \preceq n^{-1/p}$ implies $e_n(\mathrm{co}A) \preceq n^{-1/p}$ for every precompact subset A of E. Then Theorem 4.5 tells us that E must be B-convex , i.e. of some type $q > 1$. The following proposition yields an estimation of the type q that E is necessary to be of in this situation.

Proposition 4.8 *Let E be a Banach space such that for some $1 < q \leq 2$ and some σ-controlled function f we know that*

$$e_n(A) \preceq n^{-(1-1/q)} f(\log_2(n+1))$$

implies

$$e_n(\mathrm{co}A) \preceq n^{-(1-1/q)} f(\log_2(n+1))$$

for all precompact subsets A of E. Then E must be of type r for all $1 < r < q$. If we additionally have $\frac{f(x)}{f(\log_2 x)} \to 0$ for $x \to \infty$ and $1 < q < 2$, then E is even of type $q + \varepsilon$ for some $\varepsilon > 0$.

Proof: Suppose that E is not of type r. Then E is not of stable type r and hence it contains ℓ_r^n's uniformly. We let $a_n := (\log_2(n+1))^{-(1-1/q)} f(\log_2(\log_2(n+1)+1))$. Then by Example 1.2 and Theorem 4.1 we find a subset A of E such that $\varepsilon_n(A) \sim a_n$. With $m = n$ and Theorem 1.6 we also obtain

$$\begin{aligned} e_n(\mathrm{co}A) &\geq c_1\, a_n\, e_n(id : \ell_1^n \to \ell_r^n) \\ &\geq c_2\, (\log_2(n+1))^{-(1-1/q)} f(\log_2(\log_2(n+1)+1))\, n^{-(1-\frac{1}{r})}\,. \end{aligned}$$

On the other hand we know

$$e_n(\mathrm{co}A) \leq c\, n^{-(1-1/q)} f(\log_2(n+1))$$

by the assumption on E. But this gives a contradiction for large n since $r < q$. Now we additionally suppose that $\frac{f(x)}{f(\log_2 x)} \to 0$ for $x \to \infty$. By Theorem 1.14 it suffices to prove that E is of stable type q. Let us assume that E is not, then analogously to the above reasoning with $q = r$ and the same sequence (a_n), we find a subset A of E with $\varepsilon_n(A) \sim a_n$ and

$$\begin{aligned} e_n(\mathrm{co}A) &\geq c_1\, a_{n^2}\, e_n(id : \ell_1^{n^2} \to \ell_q^{n^2}) \\ &\geq c_2\, f(\log_2(\log_2(n^2+1)+1))\, n^{-(1-\frac{1}{q})}\,. \end{aligned}$$

But since we also know $e_n(\mathrm{co}A) \leq c\, n^{-(1-1/q)} f(\log_2(n+1))$, we obtain a constant $c_3 > 0$ such that

$$\begin{aligned} f(\log_2(\log_2(n+1)+1)) &\leq \left(\frac{\log_2(\log_2(n^2+1)+1)}{\log_2(\log_2(n+1)+1)}\right)^{\sigma} f(\log_2(\log_2(n^2+1)+1)) \\ &\leq c_3\, f(\log_2(n+1))\,, \end{aligned}$$

but this contradicts the assumption on f. ◀

4.2 The case of fast logarithmic decay

We continue our programme estimating the entropy numbers of convex hulls with the help of the inequalities of chapter 3. Our next aim is to give an analogue result to Theorem 3.2 and Corollary 3.5, which cover the case of logarithmic decay with large exponents. Moreover, we again investigate the local structure of Banach spaces in terms of entropy estimates. We begin with:

Theorem 4.9 *Let E be a Banach space of entropy type $q \in (1, 2]$. Then for all $p \in (0, q')$ and all σ-controlled functions $f : [0, \infty) \to (0, \infty)$ with $0 < \sigma < \frac{1}{p} - \frac{1}{q'}$ there exists a constant $c \geq 1$ such that for all precompact $A \subset E$ we have*

$$\sup_{k \leq n} k^{1/q'} (\log_2(k+1))^{1/p - 1/q'} f(\log_2(k+1)) \; e_k(\mathrm{co}A) \;\leq\; c\, c_A \sup_{k \leq a_n} k^{1/p} \; f(k) \; e_k(A) \; ,$$

where $a_n := n^{\frac{p}{q'(1-p\sigma)}} \log_2(n+1)$.
If E is a Hilbert space and $q = 2$, this is also true for the Gelfand widths $c_n(\mathrm{co}A)$.

Proof: We define $C_n := \sup_{k \leq a_n} k^{1/p} \; f(k) \; e_k(A)$. Using Theorem 3.2 we then obtain analogously to Theorem 4.2:

$$n^{1/q'} \; (\log_2(n+1))^{1/p - 1/q'} \; f(\log_2(n+1)) \; e_n(T'_A : E' \to \ell_\infty(A)) \;\leq\; c_1 \; c_A \; C_n \; ,$$

where $c_1 > 0$ is a suitable constant only depending on p, q, f and E. Thus, for $\delta = 2/q' + 1/p + \sigma$ we receive

$$\begin{aligned}
& n^\delta \; e_n(\mathrm{co}A) \\
\leq \;& c_2 \sup_{k \leq n} k^\delta \; e_k(T'_A : E' \to \ell_\infty(A)) \\
\leq \;& c_3 \; c_A \sup_{k \leq n} \frac{k^\delta \; k^{-1/q'} (\log_2(k+1))^{1/q' - 1/p}}{f(\log_2(k+1))} \; C_k \\
\leq \;& c_3 \; c_A \; C_n \; \frac{(\log_2(n+1))^\sigma}{f(\log_2(n+1))} \; \sup_{k \leq n} k^{\delta - 1/q'} \; (\log_2(k+1))^{1/q' - 1/p - \sigma} \\
\leq \;& c_3 \; c_A \; C_n \; n^{\delta - 1/q'} \; (\log_2(n+1))^{1/q' - 1/p} \; f(\log_2(n+1))^{-1} \; ,
\end{aligned}$$

where c_2 is the constant appearing in Theorem 1.18 and $c_3 := c_1 \; c_2$. Hence we have

$$n^{1/q'} \; (\log_2(n+1))^{1/p - 1/q'} \; f(\log_2(n+1)) \; e_n(\mathrm{co}A) \;\leq\; c_3 \; c_A \sup_{k \leq a_n} k^{1/p} \; f(k) \; e_k(A)$$

and this yields the assertion since (a_n) is increasing. ◀

As an easy consequence we get a dualized version of Corollary 3.5:

Corollary 4.10 *Let E be a Banach space of entropy type $q \in (1,2]$. Moreover let $p \in (0,q')$ and $f : [0,\infty) \to (0,\infty)$ be a σ-controlled function. Then for all precompact subsets A of E with*

$$e_n(A) \preceq n^{-1/p} f(\log_2(n+1))$$

we have

$$e_n(\mathrm{co}A) \preceq n^{-1/q'} (\log_2(n+1))^{1/q'-1/p} f(\log_2(\log_2(n+1)+1)) .$$

This estimate is asymptotically optimal for some subset A whenever E is not of any entropy type better than q.

Proof: The proof of the estimate is analogous to Corollary 3.5. For the optimality we first observe that E cannot be of stable type q in the case of $1 < q < 2$, since otherwise E would be of some stable type $q+\varepsilon$ and hence of entropy type $q+\varepsilon$. Therefore E contains ℓ_q^n's uniformly in this case. Moreover, if $q = 2$ we know by Dvoretzky's Theorem that E contains ℓ_q^n's uniformly. Now we let

$$a_n := (\log_2(n+1))^{-1/p} f(\log_2(\log_2(n+1)+1)).$$

The sequence (a_n) is regular by Example 1.2. Hence by Theorem 4.1 we can find a subset $A \subset E$ such that $\varepsilon_n(A) \sim a_n$ and with the help of Theorem 1.6 we obtain

$$\begin{aligned} e_n(\mathrm{co}A) &\geq c_1\, a_{n^2}\, e_n(id : \ell_1^{n^2} \to \ell_q^{n^2}) \\ &\geq c_2\, a_{n^2} \left(\frac{\log_2(n+1)}{n}\right)^{1/q'} \\ &\geq c_3\, n^{-1/q'} (\log_2(n+1))^{1/q'-1/p} f(\log_2(\log_2(n+1)+1)) . \end{aligned}$$

Therefore the estimate of the corollary is asymptotically optimal for A. ◀

If E is a Hilbert space and $q = 2$, the conclusion of the above corollary also holds for the Gelfand widths $c_n(\mathrm{co}A)$. Moreover, with a result of Garnaev and Gluskin in [17] instead of Theorem 1.6 one can easily check that the resulting estimate is also asymptotically optimal.
Again, we investigate the structure of E, if a conclusion analogous to the above corollary holds. We obtain a result similar to Proposition 4.8:

Proposition 4.11 *Let E be a Banach space such that for some $1 < q \leq 2$, $0 < p < \infty$ and some σ-controlled function f we know that*

$$e_n(A) \preceq n^{-1/p} f(\log_2(n+1))$$

implies

$$e_n(\mathrm{co}A) \preceq n^{-(1-1/q)} (\log_2(n+1))^{1-1/q-1/p} f(\log_2(\log_2(n+1)+1))$$

for all precompact subsets A of E. Then E must be of type r for all $1 < r < q$.

Proof: Suppose that E is not of type r. Then E is not of stable type r and hence it contains ℓ_r^n's uniformly. We let

$$a_n := (\log_2(n+1))^{-1/p} \ f(\log_2(\log_2(n+1)+1)) \ .$$

Then we know that (a_n) is regular by Example 1.2. Hence by Theorem 4.1 we can find a subset $A \subset E$ such that $\varepsilon_n(A) \sim a_n$ and with the help of Theorem 1.6 we obtain

$$\begin{aligned} e_n(\mathrm{co}A) &\geq c_1 \ a_{n^2} \ e_n(id : \ell_1^{n^2} \to \ell_r^{n^2}) \\ &\geq c_2 \ n^{-(1-1/r)} \ (\log_2(n+1))^{1-1/r-1/p} \ f(\log_2(\log_2(n+1)+1)) \ . \end{aligned}$$

On the other hand we know

$$e_n(\mathrm{co}A) \ \leq \ c \ n^{-(1-1/q)} \ (\log_2(n+1))^{1-1/q-1/p} \ f(\log_2(\log_2(n+1)+1))$$

by the assumption on E. But this is a contradiction for large n since $r < q$. ◀

Since there exist spaces of weak type $q \in (1,2)$ which are not of type q, we cannot expect that E is even of type q under the assumption of Proposition 4.11. However, it is not clear whether E must be of entropy type q or at least of weak type q in this situation.

4.3 The case of polynomial decay

Continuing our program we finally apply Theorem 3.4, which covers the case of polynomial decay. Again we also investigate the local structure of Banach spaces with the help of estimates of entropy numbers of convex hulls receiving this time a characterization of weak type q spaces, $1 < q < 2$. We begin with:

Theorem 4.12 *Let E be a Banach space of weak entropy type $q \in (1,2]$ and $0 < p < \infty$ as well as $\gamma \geq 0$. Then there is a constant $c \geq 1$ such that for all precompact $A \subset E$ we have:*

$$\sup_{k \leq n} k^{1-1/q+1/p} \ (\log_2(k+1))^\gamma \ e_k(\mathrm{co}A) \ \leq \ c \ c_A \sup_{k \leq n^{1+\frac{p}{q'}}} k^{1/p} \ (\log_2(k+1))^\gamma \ \varepsilon_k(A) \ .$$

Proof: With the help of Theorem 3.4, the proof is a straightforward analogue to the argument used for Theorem 4.10. ◀

Corollary 4.13 *Let E be a Banach space of weak entropy type q, $1 < q \leq 2$. Moreover let $p \in (0,\infty)$ and $f : [0,\infty) \to (0,\infty)$ be a σ-controlled function. Then there is a constant $c > 0$ such that for all precompact subsets A of E with*

$$\varepsilon_n(A) \ \leq \ n^{-1/p} \ f(\log_2(n+1)) \ , \qquad n \in \mathbb{N}$$

we have

$$e_n(\text{co}A) \;\leq\; c\; c_A\; n^{-(1-1/q)-1/p}\; f(\log_2(n+1))\;, \qquad n \in \mathbb{N}.$$

This estimate is asymptotically optimal for some subset A whenever E is not of any weak entropy type better than q.

Proof: The proof of the estimate is analogous to Corollary 3.6. To see that this is asymptotically optimal, we infer as for Corollary 4.10 that E contains ℓ_q^n's uniformly. Now we let $a_n := n^{-1/p}\, f(\log_2(n+1)$. Then the sequence (a_n) is regular by Example 1.2. Hence by Theorem 4.1 we can find a subset $A \subset E$ such that $\varepsilon_n(A) \sim a_n$, and with the help of Theorem 1.6 we obtain

$$\begin{aligned} e_n(\text{co}A) \;&\geq\; c_1\; a_n\; e_n(id : \ell_1^n \to \ell_q^n) \\ &\geq\; c_2\; n^{-(1-1/q)-1/p}\; f(\log_2(n+1))\;. \end{aligned}$$

Therefore the estimate of the corollary is asymptotically optimal for A. ◄

Again one can ask which local properties E must have, if the conclusion of Theorem 4.12, resp. Corollary 4.13 holds. This time it is necessary to take care of the arising constant:

Proposition 4.14 *Let E be a Banach space such that for some $1 < q \leq 2$, $0 < p < \infty$ and some σ-controlled function f we know that*

$$\varepsilon_n(A) \;\leq\; n^{-1/p}\; f(\log_2(n+1))\;, \qquad n \in \mathbb{N}$$

implies

$$e_n(\text{co}A) \;\leq\; c\; c_A\; n^{-(1-1/q)-1/p}\; f(\log_2(n+1))\;, \qquad n \in \mathbb{N}$$

for all precompact subsets A of E and a suitable constant $c \geq 1$ independent of A and n. Then E must be of weak type q.
If f is decreasing, then E is even of weak entropy type q.

Proof: Let $T : \ell_1^m \to E$ be an arbitrary operator. We define $x_k := Te_k$ for $1 \leq k \leq m$ where (e_k) is the canonical basis of ℓ_1^m. For $A := \{x_1, \ldots, x_m\}$ we have $TB_{\ell_1^m} = \text{co}A$ and $\|T\| = \|A\|$. Moreover, we know $\varepsilon_k(A) \leq \varepsilon_1(A)$ for $1 \leq k \leq m$ and $\varepsilon_k(A) = 0$ otherwise. Therefore with $C_m := \varepsilon_1(A)\; m^{1/p}\; (f(\log_2(m+1)))^{-1}$ we obtain

$$\begin{aligned} \varepsilon_n(A) \;&\leq\; n^{-1/p}\; f(\log_2(n+1))\; \sup_{k\geq 1} k^{1/p}\; f(\log_2(k+1))^{-1}\; \varepsilon_k(A) \\ &\leq\; C_m\; n^{-1/p}\; f(\log_2(n+1))\;, \end{aligned}$$

for all $n \geq 1$. Applying the assumption we get

$$\begin{aligned} e_n(T) \;&=\; C_m\; e_n(\frac{1}{C_m}\text{co}A) \\ &\leq\; c\; C_m\; c_A\; n^{-(1-1/q)-1/p}\; f(\log_2(n+1)) \\ &\leq\; c\; \|T\|\; m^{1/p}\; (f(\log_2(m+1)))^{-1}\; n^{-(1-1/q)-1/p}\; f(\log_2(n+1))\;. \end{aligned}$$

Then in the general case we take $n = m$ and obtain that E is of weak type q by [22, Th. 1] in the case of $1 < q < 2$ and of weak type 2 by [32] in the case of $q = 2$. (cf. [22, Rem. (2), p. 424]).
If f is decreasing, there is nothing more to prove. ◄

Corollary 4.13 together with Proposition 4.14 yields an interesting characterization of weak type q, resp. weak entropy type q spaces:

Corollary 4.15 *Let E be a Banach space and $1 < q \le 2$. Then the following are equivalent:*

i) E is of weak entropy type q.

ii) For some or all $p \in (0, \infty)$ there exists a constant $c > 0$, such that for every precompact symmetric *subset $A \subset E$ with $\varepsilon_n(A) \le n^{-1/p}$, $n \ge 1$, we have*

$$e_n(\mathrm{co}A) \;\le\; c \cdot n^{-(1-1/q)-1/p} \,, \qquad n \in \mathbb{N}.$$

In particular this is equivalent to E being of weak type q in the case of $1 < q < 2$.

4.4 Remarks

Remark 4.16 Theorem 4.12 and Corollary 4.13 also hold for the Gelfand widths provided that E is a Hilbert space and $q = 2$. Moreover, we obtain a result analogously to Remark 3.8:

Let E be a $\mathcal{L}_q$-space for some $1 < q < \infty$ and $q^* := \max\{q, q'\}$. Then for all $p > 0$ with $\frac{1}{2} < \frac{1}{q^*} + \frac{1}{p}$ and every $\gamma \ge 0$ there is a constant $c \ge 1$, such that for all precompact subsets A of E we have

$$\sup_{k \le n} k^{\frac{1}{p}+\frac{1}{q^*}} \,(\log_2(k+1))^\gamma\, c_k(\mathrm{co}A) \;\le\; c\, c_A\, \|T\|_1 \sup_{k \le n^{1+\frac{p}{q^*}}} k^{\frac{1}{p}}\, (\log_2(k+1))^\gamma\, \varepsilon_k(A) \,.$$

To prove this we just have to carefully repeat the proof of Theorem 3.4 in a 'dual version' using inequality (1.1). As a direct consequence we obtain:

Let E be a $\mathcal{L}_q$-space for some $1 < q < \infty$ and $q^* := \max\{q, q'\}$. Moreover, let $p \in (0, \infty)$ with $\frac{1}{2} < \frac{1}{q^*} + \frac{1}{p}$ and f be a σ-controlled function. Then for all precompact subsets A of E with

$$\varepsilon_n(A) \;\preceq\; n^{-1/p}\, f(\log_2(n+1))$$

we have

$$c_n(\mathrm{co}A) \;\preceq\; n^{-(1-1/q)-1/p}\, f(\log_2(n+1)) \,.$$

At least for $1 < q \le 2$ and p and f as above this estimate is asymptotically optimal for some corresponding subset A of E.

Note that the subset A for which the estimate is optimal can be constructed analogously to Corollary 4.13. To estimate $c_n(\mathrm{co}A)$ from below we then use a result of Garnaev and Gluskin on the behaviour of $c_n(id : \ell_1^n \to \ell_p^n)$, $1 < p \le 2$, in [17].

Remark 4.17 Analogously to Remark 3.7 one might ask what happens for $(e_n(\mathrm{co}A))$, if we know that $(\varepsilon_n(A)) \in \ell_{p,r}$. Again, we can only give a partial answer:

Let E be a Banach space of weak entropy type q and let $0 < p, r < \infty$. Then there is a constant $c_{p,r} > 0$ such that for every precompact subset A of E with

$$2^{1/p}\ e_{n+1}(A) \ \le\ e_n(A)\ , \qquad\qquad n \ge 1 \tag{4.2}$$

we have

$$\sum_{k=1}^{n} \left(k^{1/p+(1-1/q)-1/r}\ e_k(\mathrm{co}A)\right)^r \ \le\ c_{p,r}\ c_A \sum_{k=1}^{n^{1+p/q'}} \left(k^{1/p-1/r}\ \varepsilon_k(A)\right)^r\ .$$

In particular, $(\varepsilon_n(A)) \in \ell_{p,r}$ together with condition (4.2) implies $(e_n(\mathrm{co}A)) \in \ell_{s,r}$ where $\frac{1}{s} = \frac{1}{p} + 1 - \frac{1}{q}$.

The proof is based on Remark 3.7. If we consider subsets of the form $A = \{\ \sigma_n x_n \mid n \in \mathbb{N}\ \}$ with (σ_n) decreasing and $\|x_n\| = 1$ and additionally replace $\varepsilon_n(A)$ by σ_n, we can drop condition (4.2) using interpolation instead. For this we also refer to [5, Th. 1].

Remark 4.18 Let E be a Banach space of entropy type q and suppose that we have a subset A of E with $(e_n(A)) \in \ell_{q',\infty}$. In [10] Carl, Kyrezi and Pajor asked for an asymptotically optimal estimate of $(e_n(\mathrm{co}A))$. Using inequality (3.10) we obtain

$$\sup_{k\le n} k^{1/q'}\ (\log_2(k+1))^{-(1+1/q')}\ e_k(\mathrm{co}A)\ \le\ c\ c_A\ \sup_{k\le n} k^{1/q'}\ e_k(A)$$

for every precompact subset A of E, but again we conjecture that one can drop the extra $\log_2$-factor. Moreover, if we know that $(e_n(K))$ decreases slightly faster than $n^{-1/q'}$, say $e_n(K) \preceq n^{-1/q'}\ (\log_2(k+1))^{-\gamma}$ for some $\gamma > 1 + 1/q'$, we can apply

$$\sup_{k\le n} k^{\frac{1}{q}'} (\log_2(\log_2(k+1)+1))^{\gamma-(1+\frac{1}{q}')}\ e_k(\mathrm{co}A)\ \le\ c\ c_A\ \sup_{k\le n} k^{\frac{1}{q}'} (\log_2(k+1))^{\gamma}\ e_k(A)$$

which can be derived analogously to Theorem 3.2. One easily checks that this is asymptotically optimal apart from the factor $(\log_2(\log_2(k+1)+1))^{-(1+\frac{1}{q})}$. However, an optimal result is still missing apart from the following estimate which is due to Li and Linde in [28, Th. 5.1]:

Let H be a Hilbert space and $A = \{x_1, x_2, \dots\}$ be a countable subset of H with

$$\|x_n\| \preceq (\log_2(n+1))^{-1/2} (\log_2(\log_2(n+1)+1))^\gamma .$$

Then we have

$$e_n(\mathrm{co}A) \preceq \begin{cases} n^{-1/2} (\log_2(n+1))^\gamma & \text{if} \quad \gamma \geq 0 \\ n^{-1/2} (\log_2(\log_2(n+1)+1))^\gamma & \text{if} \quad \gamma < 0 . \end{cases}$$

Remark 4.19 It is an interesting phenomenon that covering properties of convex sets characterize B-convexity and weak type since the latter ones are purely local properties while the considered convex sets cannot be reduced to finitely many dimensions in general. In particular, this is true for Theorem 4.5, since subsets A with regular sequences $(e_n(\mathrm{co}A))$ do not span a finite dimensional subspace.
The nature of Corollary 4.15 is slightly different: Roughly speaking, it states that for the definition of weak entropy type q, which is equivalent to weak type q for $1 < q < 2$, we can also use symmetric subsets A of E with $\varepsilon_n(A) \preceq n^{-1/p}$ instead of finite subsets A. Note that the former ones include the latter ones.

Remark 4.20 The problem of estimating $e_n(\mathrm{co}A)$ in terms of $\varepsilon_n(A)$ was initially considered by Dudley in [16]. He proved a weaker form of Corollary 4.13 for Hilbert spaces and used it together with an entropy condition of Pollard to determine whether a class of measurable functions is a universal Donsker class.
Ball and Pajor [1] as well as Carl [7] improved his result in the Hilbert space case. The former ones also asked for *inequalities* between $e_n(\mathrm{co}A)$ and $\varepsilon_n(A)$ which produce both known and new results.
Finally, Carl, Kyrezi and Pajor [10] generalized the case of polynomial decay to Banach spaces of weak type q receiving a version of Corollary 4.13 without the factor $f(\log_2(n+1))$. Moreover, they showed statements for $\varepsilon_n(A) \leq (\log_2(n+1))^{-1/p}$ similar to those of the Corollaries 4.3 and 4.10 and proved a version of inequality (4.1).

Chapter 5

More on the entropy of $C(K)$-valued operators and its applications

The last chapter is devoted to three topics. Firstly, we investigate the local structure of Banach spaces in terms of entropy estimates of 1-Hölder-continuous operators. Moreover, we show that the inequalities of chapter 3 produce asymptotically optimal results. For both we need the techniques used for similar questions in the previous chapter.
The third aim is to give another application of Theorem 3.1: We show how the Tichomirov numbers of a compact operator $T : E \to F$ can be estimated by its entropy numbers provided that one of the spaces is a Hilbert space. Some remarks on the duality problem of entropy numbers are added. The proof is based on the observation that every compact operator $T : E \to F$ shares its entropy numbers with a suitable 1-Hölder-continuous operator $S : E \to \ell_\infty((K,d))$ where K is essentially a subset of $B_{F'}$ and the metric d is defined with the help of the dual operator $T' : F' \to E'$. This is the major aim of the following lemma:

Lemma 5.1 *Let $T : E \to \ell_\infty(A)$ be an arbitrary operator. Then there is an equivalence relation $\sim$ on A, a metric d on $K := A_{/\sim}$ and a 1-Hölder-continuous operator $S : E \to \ell_\infty(K,d)$ with $|S|_1 = 1$ such that*

i) $\|S\| = \|T\|$ *and* $s_n(T) \leq s_n(S) \leq 2\ s_n(T)$ *for* $s \in \{c,e,t\}$ *and all* $n \geq 1$.

ii) $\varepsilon_n(K) = \varepsilon_n((\{T'e_s \mid s \in A\}, \|.\|))$ *for all* $n \geq 1$, *where* $(e_s)_{s\in A}$ *is the canonical basis of* $\ell_1(A)$.

In particular if T is compact, then (K,d) is precompact.

Proof: We let $s \sim t$ if $T'e_s = T'e_t$ and define $d([s],[t]) := \|T'(e_s - e_t)\|$ on K. Clearly, d is a well-defined metric. For $x \in E$ and $s \in A$ we let $Sx([s]) := Tx(s)$.

We observe that $Sx([s]) = \langle x, T'e_s\rangle$, in particular S is well-defined and 1-Hölder-continuous. Moreover, we let

$$\begin{aligned} I : \ell_\infty(K) &\to \ell_\infty(A) \\ f &\mapsto (s \mapsto f([s])) \ . \end{aligned}$$

Then one easily checks that I is a metric injection and $T = IS$. Therefore $i)$ is proved. To see $ii)$ we observe that $[s] \mapsto T'e_s$ is a bijective map between K and $\{T'e_s \mid s \in A\}$ which preserves distances. ◀

5.1 On the local structure of Banach spaces in terms of entropy estimates for $C(K)$-valued operators

Similarly to chapter 4 we investigate the local structure of an arbitrary Banach space E under the assumption of known estimates of entropy numbers of 1-Hölder-continuous operators $T : E \to C(K)$. We begin with a characterization of B-convexity similar to Theorem 4.5:

Theorem 5.2 *The following statements on a Banach space E are equivalent:*

i) E is not B-convex

ii) For all regular sequences (a_n) there exists a precompact metric space (K, d) and a 1-Hölder-continuous operator $T : E \to \ell_\infty(K)$ with

$$\varepsilon_n(K) \sim a_n \sim e_n(T) \ .$$

This Theorem in particular states that B-convex Banach spaces are the only ones for which the estimate (3.1) can be improved. One might think that this theorem is a direct consequence of Theorem 4.5. However, since there is no positive answer to the duality problem of entropy numbers in non B-convex Banach spaces yet, we have to repeat parts of the proof:

Proof: Suppose that E is B-convex. Let $a_n := (\log_2(\log_2(n+1)+1))^{-1}$. Then Corollary 3.5 tells us that for every precompact metric space (K, d) with $\varepsilon_n(K) \sim a_n$ and all 1-Hölder-continuous operators $T : E \to \ell_\infty(K)$ we have $e_n(T) \preceq a_{2^n}$. Hence $ii)$ does not hold.
Now suppose E is not B-convex. Then E' is not B-convex either and therefore it contains ℓ_1^n's uniformly by Pisier's Theorem. Hence without loss of generality there are subspaces $E_n \subset E'$ and isomorphisms $T_n : E_n \to \ell_1^{2^n}$ with $\|T_n\| = 1$ and $\|T_n^{-1}\| \le 2$.

Let I_n be the embedding of E_n into E'. Denoting by $e_1^{(n)}, \dots, e_{2^n}^{(n)}$ the canonical basis of $\ell_1^{2^n}$ we define

$$A_n := \{a_{2^n} \, I_n T_n^{-1} e_j^{(n)} \mid 1 \leq j \leq 2^n\} .$$

Moreover we let

$$A := \bigcup_{n=0}^{\infty} A_n \cup \{0\} .$$

Since this is exactly the same construction used in the proof of Theorem 4.1, we already know $\varepsilon_n(A) \sim a_n$. With the help of the operator $T_A : \ell_1(A) \to E'$ defined by $T_A e_t := t$ on the canonical basis of $\ell_1(A)$, we let $T := (T'_A)_{|E} : E \to \ell_\infty(A)$. Then T is 1-Hölder-continuous and by inequality (3.1) we already know $e_n(T) \preceq a_n$. To estimate from below, we also need the projections

$$\begin{aligned} P_n : \ell_\infty(A) &\to \ell_\infty^{2^n} \\ f &\mapsto (f(a_{2^n} I_n T_n^{-1} e_1^{(n)}), \dots, f(a_{2^n} I_n T_n^{-1} e_{2^n}^{(n)})) . \end{aligned}$$

Then for $x \in E$, $1 \leq i \leq 2^n$ and $t := a_{2^n} I_n T_n^{-1} e_i^{(n)}$ we have

$$\langle P_n T x, e_i^{(n)} \rangle = \langle Tx, e_t \rangle = \langle x, T_A e_t \rangle = a_{2^n} \langle x, I_n T_n^{-1} e_i^{(n)} \rangle = a_{2^n} \langle (I_n T_n^{-1})' x, e_i^{(n)} \rangle ,$$

i.e. $P_n T = a_{2^n} ((I_n T_n^{-1})')_{|E}$. Moreover, $(I'_n)_{|E}$ is a metric surjection and $(T_n T_n^{-1})' = id_{\ell_\infty^{2^n}}$. Hence for $k := \lfloor \log_2 n \rfloor$ we obtain

$$\begin{aligned} e_n(T) &\geq e_n(P_{k+1} T) \\ &= a_{2^{k+1}} \; e_n((T_{k+1}^{-1})') \\ &\geq a_{2^{k+1}} \; e_n(id : \ell_\infty^{2^{k+1}} \to \ell_\infty^{2^{k+1}}) \\ &\geq c \; a_n , \end{aligned}$$

where $c > 0$ is a suitable constant independent of n. ◀

Analogously to Theorem 4.5 we cannot simplify condition *ii)* to

ii') There is a compact metric space (K, d) and a 1-Hölder-continuous operator $T : E \to C(K)$ such that $(\varepsilon_n(K))$ is regular and

$$\varepsilon_n(K) \sim e_n(T) .$$

The construction of a counterexample is analogous to example 4.6, therefore we drop it.

We now investigate the local structure of a Banach space E under the assumption of known estimates for 1-Hölder-continuous operators $T : E \to C(K)$. The results are similar to the analogue considerations of the previous chapter. We begin with:

Proposition 5.3 *Let E be a Banach space such that for some $2 \le q < \infty$ and some σ-controlled function f we know that for every precompact metric space (K,d) with*

$$e_n(K) \;\preceq\; n^{-1/q}\; f(\log_2(n+1))$$

we have

$$e_n(T) \;\preceq\; n^{-1/q}\; f(\log_2(n+1))$$

for all 1-Hölder-continuous operators $T : E \to \ell_\infty(K)$. Then E must be B-convex and of cotype r for all $q < r < \infty$.
If we additionally have $\frac{f(x)}{f(\log_2 x)} \to 0$ for $x \to \infty$ and $2 < q < \infty$, then E is even of cotype r for some $2 < r < q$.

Proof: First of all we observe that E is B-convex by Theorem 5.2. Now let A be a precompact subset of E' with

$$e_n(A) \;\le\; n^{-1/q}\; f(\log_2(n+1)) \; .$$

The operator $T'_A : E'' \to \ell_\infty(A)$ is 1-Hölder-continuous. Moreover, for the operator $S := (T'_A)_{|E} : E \to \ell_\infty(A)$ we have $(S')_{|\ell_1(A)} = T_A$ since

$$\langle S'y, x\rangle \;=\; \langle y, Sx\rangle \;=\; \langle y, (T'_A)_{|E}x\rangle \;=\; \langle y, T'_A x\rangle \;=\; \langle T_A y, x\rangle$$

for all $y \in \ell_1(A)$ and $x \in E$. Now our assumption applied to S yields

$$e_n(S) \;\le\; c\, n^{-1/q}\; f(\log_2(n+1)) \; .$$

Hence with the help of Corollary 1.19, we find a constant $c_1 \ge 1$ such that we finally obtain

$$e_n(\mathrm{co}A) \;=\; e_n(T_A) \;=\; e_n((S')_{\ell_1(A)}) \;\le\; e_n(S') \;\le\; c_1\; n^{-1/q}\; f(\log_2(n+1)) \; .$$

But then Proposition 4.8 tells us that E' must be of entropy type r for all $1 < r < q'$, resp. for some $q' < r < 2$, and therefore we get the assertions by Proposition 1.27. ◀

With the same arguments we obtain a dualized version of Proposition 4.11:

Proposition 5.4 *Let E be a Banach space such that for some $2 \le q < \infty$, $0 < p < \infty$ and some σ-controlled function f we know that for every precompact metric space (K,d) with*

$$e_n(K) \;\preceq\; n^{-1/p}\; f(\log_2(n+1))$$

we have

$$e_n(T) \;\preceq\; n^{-1/q}\; (\log_2(n+1))^{1/q-1/p}\; f(\log_2(\log_2(n+1)+1))$$

for all 1-Hölder-continuous operators $T : E \to \ell_\infty(K)$. Then E must be B-convex and of cotype r for all $q < r < \infty$.

In the case of polynomial decay we are also able to give a dualized version of Proposition 4.14, but this time we need some further work for the proof:

Proposition 5.5 *Let E be a Banach space such that for some $2 \le q < \infty$, $0 < p < \infty$ and some σ-controlled function f we know that for every precompact metric space (K, d) with*

$$\varepsilon_n(K) \;\le\; \rho \, n^{-1/p} \, f(\log_2(n+1)) \;, \qquad n \in \mathbb{N}$$

we have

$$e_n(T) \;\le\; c \, \rho \, c_K \, \|T\|_1 \, n^{-1/q-1/p} \, f(\log_2(n+1)) \;, \qquad n \in \mathbb{N}$$

for all 1-Hölder-continuous operators $T : E \to \ell_\infty(K)$ and a suitable constant $c \ge 1$ independent of K, T and n. Then E is B-convex and of weak cotype q.
If f is monotonously decreasing then E is even of weak entropy cotype q.

Proof: First of all we observe that E is B-convex by Theorem 5.2. Now we take an arbitrary subset $A = \{x_1, \ldots, x_m\}$ of E'. Without loss of generality we may assume $\|A\| = 1$. For the operator $T := (T'_A)_{|E} : E \to \ell_\infty^m$ we then have $\|T\| = 1$ and $T' = T_A$. By Lemma 5.1 we find a 1-Hölder-continuous operator $S : E \to \ell_\infty(K)$ with $\|S\|_1 = 1$ and $e_n(T) \le e_n(S)$. Additionally, we know

$$\varepsilon_1(K) \;\le\; 2 \; \|T\| \;=\; 2$$

and $|K| \le m$. In particular we have $\varepsilon_k(K) = 0$ for all $k > m$. Moreover there is a constant $c > 0$ such that

$$\begin{aligned} \varepsilon_n(K) \;&\le\; n^{-1/p} \, f(\log_2(n+1)) \, \sup_{k \ge 1} k^{1/p} \, f(\log_2(k+1))^{-1} \, \varepsilon_k(K) \\ &\le\; c \, \varepsilon_1(K) \, m^{1/p} \, f(\log_2(m+1))^{-1} \, n^{-1/p} \, f(\log_2(n+1)) \end{aligned}$$

and therefore our assumption yields

$$\begin{aligned} e_m(T) \;&\le\; e_m(S) \\ &\le\; c \, c_K \, \varepsilon_1(K) \, \|S\|_1 \, m^{1/p} \, f(\log_2(m+1))^{-1} \, n^{-1/q-1/p} \, f(\log_2(n+1)) \\ &\le\; 2 \, c \; \|T\| \; m^{1/p} \, f(\log_2(m+1))^{-1} \, n^{-1/q-1/p} \, f(\log_2(n+1)) \;. \end{aligned}$$

Hence with Theorem 1.18 we obtain

$$\begin{aligned} e_m(\mathrm{co}A) \;&=\; e_m(T') \\ &\le\; c_1 \; \|T\| \; m^{1/p} \, f(\log_2(m+1))^{-1} \, n^{-1/q-1/p} \, f(\log_2(n+1)) \end{aligned}$$

for some constant $c_1 > 0$ independent of A, m and n. Then in the general case we take $n = m$ and obtain that E' is of weak type q' by [22, Th. 1] in the case of $2 < q < \infty$ and of weak type 2 by [32] in the case of $q = 2$ (cf. [22, Rem. (2), p. 424]). If f is decreasing, E' is of weak entropy cotype q'. In both cases this is equivalent to the assertion. ◀

Remark 5.6 For the proof of Proposition 5.4 we can also use the idea of the above proposition, if we take care of the constants arising in the estimates of the assumption. But in this case we only obtain

$$e_n(T_A) \leq c\, n^{-1/q}\ (\log_2(n+1))^{1/q}\ \ \|T\|\ ,$$

which indicates that the result of Proposition 5.4 might be the best possible. Estimating $e_n(T_A)$ with the help of the assumptions of Proposition 5.3 one even gets worse inequalities. Therefore in this case the corresponding result may also be the best.

Corollary 3.6 together with Proposition 5.5 yields an interesting characterization of Banach spaces having weak entropy cotype q, resp. weak cotype q. Note that the following corollary also clarifies the 'local estimate' used in [12, Th. 5.10.1.] for $q \in (1,2)$.

Corollary 5.7 *Let E be a Banach space and $2 \leq q < \infty$. Then the following are equivalent:*

i) E is of weak entropy cotype q.

ii) For some or all $p \in (0,\infty)$ there exists a constant $c > 0$ such that for all compact metric spaces (K,d) with $\varepsilon_1(K) = 1$ and $\varepsilon_n(K) \leq n^{-1/p}$ we have

$$e_n(T)\ \leq\ c \cdot \|T\|_1 \cdot n^{-(1-1/q)-1/p}$$

for all 1-Hölder-continuous operators $T : E \to C(K)$.

In particular this is equivalent to E being B-convex and of weak cotype q in the case of $2 < q < \infty$.

5.2 On the optimality of the estimates proved in chapter 3

In this section we show that the estimates of the Corollaries 3.5 and 3.6 cannot be asymptotically improved under natural conditions on the domain E. Although the proofs are straightforward modifications of corresponding results in the previous chapter, we have to reason carefully since we need to estimate $e_n(T)$ from *below* with the help of known $(e_n(T'))$. We begin with the case of 'essentially slow logarithmic decay':

Proposition 5.8 *Let E be an infinite dimensional Banach space of entropy cotype $q \in [2,\infty)$. Moreover let $q < p \leq \infty$ and f be a σ-controlled function. Then there is a precompact metric space (A,d) with*

$$e_n(A)\ \sim\ n^{-1/p}\ f(\log_2(n+1))$$

and a 1-Hölder-continuous operator $T : E \to \ell_\infty(A)$ *with*

$$e_n(T) \;\sim\; n^{-1/p}\; f(\log_2(n+1)) \;.$$

In particular this is true if E *is a* $\mathcal{L}_{\tilde{q}}$*-space,* $1 < \tilde{q} < \infty$ *and* $q = \max\{2, \tilde{q}\}$*.*

Proof: Since E' contains ℓ_2^n's uniformly by Dvoretzky's Theorem, we can construct a precompact subset A of E' with

$$e_n(A) \;\sim\; n^{-1/p}\; f(\log_2(n+1))$$

using the idea of Theorem 4.1. Then Corollary 4.3 tells us that

$$e_n(\mathrm{co}A) \;\sim\; n^{-1/p}\; f(\log_2(n+1)) \;.$$

Now we consider the operator $T := (T'_A)_{|E} : E \to \ell_\infty(A)$ analogously to Proposition 5.3. Since $e_n(T_A) \sim n^{-1/p}\; f(\log_2(n+1))$ we first have

$$e_n(T) \;=\; e_n((T'_A)_{|E}) \;\leq\; e_n(T'_A) \;\sim\; n^{-1/p}\; f(\log_2(n+1))$$

by Corollary 1.19. Hence we get $e_n(T') \preceq n^{-1/p}\; f(\log_2(n+1))$ by the same corollary. Moreover we have

$$n^{-1/p}\; f(\log_2(n+1)) \;\sim\; e_n(T_A) \;=\; e_n(T'_{|\ell_1(A)}) \;\leq\; e_n(T')$$

and therefore we know $e_n(T') \sim n^{-1/p}\; f(\log_2(n+1))$. But then Corollary 1.19 tells us that

$$e_n(T) \;\sim\; n^{-1/p}\; f(\log_2(n+1)) \;. \blacktriangleleft$$

For the case of essentially fast logarithmic decay we have a similar result. However, we have to ensure that the known entropy cotype of the domain is best possible:

Proposition 5.9 *Let* E *be a Banach space of entropy cotype* $q \in [2, \infty)$*, which is not of any entropy cotype* $q - \varepsilon$*. Moreover let* $0 < p < q$ *and* f *be a* σ*-controlled function. Then there is a precompact metric space* (A, d) *with*

$$e_n(A) \;\sim\; n^{-1/p}\; f(\log_2(n+1))$$

and a 1-Hölder-continuous operator $T : E \to \ell_\infty(A)$ *with*

$$e_n(T) \;\sim\; n^{-1/q}\; (\log_2(n+1))^{1/q-1/p}\; f(\log_2(\log_2(n+1)+1)) \;.$$

In particular this is true if E *is a* $\mathcal{L}_{\tilde{q}}$*-space,* $1 < \tilde{q} < \infty$ *and* $q = \max\{2, \tilde{q}\}$*.*

Proof: By Proposition 1.27 the dual E' is of entropy type q' but not of any entropy type $r > q'$. Therefore by Corollary 4.10 there is a subset A of E' with

$$e_n(A) \ \sim \ n^{-1/p} \, f(\log_2(n+1))$$

and

$$e_n(\mathrm{co}A) \ \sim \ n^{-1/q} \, (\log_2(n+1))^{1/q-1/p} \, f(\log_2(\log_2(n+1)+1)) \ .$$

Now we obtain the assertion by the same reason as in Proposition 5.3. ◄

Analogously, we obtain that the estimate of Corollary 3.6 yields asymptotically optimal estimates.

Proposition 5.10 *Let E be a Banach space of weak entropy cotype $q \in [2, \infty)$ which is not of any weak entropy cotype $q - \varepsilon$. Moreover let $0 < p < \infty$ and f be a σ-controlled function. Then there is a precompact metric space (A, d) with*

$$\varepsilon_n(A) \ \sim \ n^{-1/p} \, f(\log_2(n+1))$$

and a 1-Hölder-continuous operator $T : E \to \ell_\infty(A)$ with

$$e_n(T) \ \sim \ n^{-1/p-1/q} \, f(\log_2(n+1)) \ .$$

In particular this is true if E is a $\mathcal{L}_{\tilde{q}}$-space, $1 < \tilde{q} < \infty$ and $q = \max\{2, \tilde{q}\}$.

Remark 5.11 The propositions of this section construct a suitable precompact metric space for which there is a 1-Hölder-continuous operator such that the estimates are asymptotically optimal. It is also interesting, whether there is such an operator if we fix *both* E and (A, d).

5.3 An inverse Carl-inequality

In this last section we prove an 'inverse' form of Theorem 1.4 for compact operators $T : E \to F$ provided that one of the spaces is a Hilbert space. Using Lemma 5.1 it turns out that this is a direct consequence of Theorem 3.1 and Remark 3.8.

Theorem 5.12 *Let E and F be Banach spaces and one of them a Hilbert space. Then for every $2 < p < \infty$ there exists a constant $c_p > 1$ such that for every compact operator $T : E \to F$ we have*

$$\frac{1}{c_p} \sup_{k \le n} k^{1/p} \, e_k(T) \ \le \ \sup_{k \le n} k^{1/p} \, t_k(T) \ \le \ c_p \ \sup_{k \le n} k^{1/p} \, e_k(T) \ .$$

Proof: First we assume that E is a Hilbert space. We consider the operator $S := IT$, where $I : F \to \ell_\infty(B_{F'})$ denotes the canonical injection. By Lemma 5.1 there is a precompact metric space K and a 1-Hölder-continuous operator $R : E \to \ell_\infty(K)$ with

$$t_n(S) \;\leq\; t_n(R) \;\leq\; 2\, t_n(S)\ .$$

Moreover we have

$$e_n(K) \;=\; e_n(\{S'e_{y'}\ y' \in B_{F'}\}) \;=\; e_n(T'I') \;=\; e_n(T')\ .$$

Therefore we obtain by Remark 3.8 and Theorem 1.18:

$$\begin{aligned}\sup_{k\leq n} k^{1/p}\ t_k(T) \;&\leq\; \sup_{k\leq n} k^{1/p}\ t_k(R)\\ &\leq\; c\ \sup_{k\leq n} k^{1/p}\ e_k(K)\\ &\leq\; \tilde{c}_p\ \sup_{k\leq n} k^{1/p}\ e_k(T)\ .\end{aligned}$$

The inverse inequality follows by Theorem 1.4.
If F is a Hilbert space, we consider the operator $S := IT'$ where $I : E' \to \ell_\infty(B_E)$ is the canonical injection. Repeating the above proof we are now using $t_n(T) = t_n(T')$ instead of Theorem 1.18. ◀

As an example of the consequences of the above theorem we obtain:

Corollary 5.13 *Let H be a Hilbert space, $p \in (2,\infty)$ and (a_n) be a regular sequence with $a_n \leq 2^{1/p}a_{2n}$. Then for every finite measure μ and every compact operator $T : H \to L_\infty(\mu)$ we have*

$$d_n(T) \;\preceq\; a_n \qquad \textit{if and only if} \qquad e_n(T) \;\preceq\; a_n$$

and

$$d_n(T) \;\sim\; a_n \qquad \textit{if and only if} \qquad e_n(T) \;\sim\; a_n\ .$$

This is also true for every compact operator $T : L_1(\mu) \to H$, if we replace the Kolmogorov numbers by the Gelfand numbers.

Proof: Since $L_\infty(\mu)$ has the extension property (cf. [15, Th. 4.14]), we have $d_n(T) = t_n(T)$ by [12, Prop. 2.3.3.]. Hence using the trick of Corollary 1.19, we obtain the assertion.
If we consider an operator $T : L_1(\mu) \to H$ we use $c_n(T) = d_n(T' : H \to L_\infty(\mu))$ and Corollary 1.19. ◀

Remark 5.14 There is a similar result to Theorem 5.12 proven by Pajor and Tomczak-Jaegermann in [33]:

Let E be a Banach space and H be a Hilbert space. Then for every $p \in (0,2)$ there is a constant $c_p > 1$ such that for every compact operator $T : E \to H$ we have

$$\frac{1}{c_p} \sup_{k \geq 1} k^{1/p} \ e_k(T) \ \leq \ \sup_{k \geq 1} k^{1/p} \ c_k(T) \ \leq \ c_p \ \sup_{k \geq 1} k^{1/p} \ e_k(T) \ .$$

Note that this inequality compares Gelfand numbers instead of Tichomirov numbers with entropy numbers. Hence this result is stronger in that sense. However, they compared suprema build by the *whole* sequences $(c_n(T))$ and $(e_n(T))$, while we have compared *finitely* many Tichomirov numbers with *finitely* many entropy numbers. Therefore our result yields much more information on the asymptotic behaviour of the involved sequences. Corollary 5.13 illustrates this.
Another similar result was shown by Carl in [6]:

Let E be a Banach space of type 2 and F be a Banach space such that F' is also of type 2. Then for all $0 < p < \infty$ and every compact operator $T : E \to F$ we have

$$\sup_{k \leq n} k^{1/p} \ e_k(T) \ \sim \ \sup_{k \leq n} k^{1/p} \ c_k(T) \ \sim \ \sup_{k \leq n} k^{1/p} \ d_k(T) \ .$$

Remark 5.15 Another idea to apply Theorem 3.1 is to use it in the original form. We then estimate $e_n(T)$ by some $e_n(T')$ and conversely. More precisely we obtain:

Let F' be a Banach space of entropy cotype q. Then for every $q < p < \infty$ there exists a constant c_p such that for every compact operator $T : E \to F$ we have

$$\sup_{k \leq n} k^{1/p} \ e_k(T') \ \leq \ c_p \ \sup_{k \leq n} k^{1/p} \ e_k(T) \ .$$

and

Let E be a Banach space of entropy cotype q. Then for every $q < p < \infty$ there exists a constant c_p such that for every compact operator $T : E \to F$ we have

$$\sup_{k \leq n} k^{1/p} \ e_k(T) \ \leq \ c_p \ \sup_{k \leq n} k^{1/p} \ e_k(T') \ .$$

However, there are two disadvantages of this ansatz: First of all, nontrivial entropy cotype implies B-convexity. Hence the above statements are covered by Theorem 1.18. Moreover, the common way to show that a Banach space is of entropy cotype q is to check that its dual is of (weak) type q' and then to apply Theorem 1.23 together with Proposition 1.27. But the latter one is based on Theorem 1.18!

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Selbständigkeitserklärung

Ich erkläre, daß ich die vorliegende Arbeit selbständig und nur unter Verwendung der angegebenen Hilfsmittel und Literatur angefertigt habe.

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